COORDINATES FOR ANALYTIC OPERATOR ALGEBRAS

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1. Introduction. Let M be a σ-finite von Neumann algebra and $\alpha = \{\alpha_t\}_{t\in A}$ be a representation of a compact abelian group A as *-automorphisms of M. Let Γ be the dual group of A and suppose that Γ is totally ordered with a positive semigroup $\Sigma \subseteq \Gamma$. The *analytic algebra* associated with α and Σ is

$$M^\alpha(\Sigma) = \{a \in M : \mathrm{sp}_\alpha(a) \subseteq \Sigma\},$$

where $\mathrm{sp}_\alpha(a)$ is Arveson's spectrum. These algebras were studied (also for A not necessarily compact) by several authors starting with Loebl and Muhly [**10**].

In the case where the fixed point algebra

$$M_0 = \{a \in M : \alpha_t(a) = a \text{ for every } t \text{ in } A\},$$

is a Cartan subalgebra of M it was shown in [**13**] that one can construct a "system of coordinates" for M and use it to study the σ-weakly closed M_0-bimodules of M. Using this analysis one can identify the σ-weakly closed ideals of $M^\alpha(\Sigma)$, the algebras that lie between the algebra $M^\alpha(\Sigma)$ and M, and other $M^\alpha(\Sigma)$-bimodules. These results were used to study isomorphisms between two such algebras.

In the present paper we do not assume that M_0 is a Cartan subalgebra or even abelian. We show (Section 2) that one can construct a "system of coordinates" for M (namely, represent each operator T in M as a "generalized matrix" $\{T(x, y) : (x, y) \in R\}$, where R is an equivalence relation on some measure space (X, μ)).

We use this representation to characterize the σ-weakly closed M_0-bimodules of M. If $M \cap Z(M_0)' \subseteq M_0$ (where $Z(M_0)$ is the center of M_0), then it is shown that for every such bimodule $\mathcal{U}$ there is a Borel subset $Q \subseteq R$ such that

$$\mathcal{U} = \{T \in M : T(x, y) = 0 \text{ for } (x, y) \text{ not in } Q\}.$$

In Section 4 we use this analysis to study M-reflexivity of M_0-bimodules. Among other things we show that α is inner if and only if $M \cap Z(M_0)' \subseteq M_0$ and every σ-weakly closed M_0-bimodule is M-reflexive.

Section 5 deals with isomorphisms $\varphi : M^\alpha(\Sigma_1) \to B^\eta(\Sigma_2)$. It is proved (Theorem 5.1) that if $M \cap Z(M_0)' \subseteq M_0$ and $B \cap Z(B_0)' \subseteq B_0$ and ψ is an algebraic isomorphism such that $\psi(a^*) = \psi(a)^*$ for $a \in M_0$, then there is an isomorphism of the equivalence relation R_1 (associated with (M, α)) onto R_2 (associated with (B, η)) that carries P_1 onto P_2. Here P_1 and P_2 are the support sets of $M^\alpha(\Sigma_1)$ and $B^\eta(\Sigma_2)$; namely,

$$M^\alpha(\Sigma_1) = \{T \in M : T(x, y) = 0 \text{ for } (x, y) \text{ not in } P_1\},$$

$$B^\eta(\Sigma_2) = \{T \in B : T(x, y) = 0 \text{ for } (x, y) \text{ not in } P_2\}.$$

This result is related to the results of [**13**, Section 5], [**11**] and [**12**].

2. Preliminaries. Let M be a σ-finite von Neumann algebra acting on a Hilbert space H and let α be a σ-weakly continuous representation of a compact abelian group A as *-automorphisms of M. Write Γ for the dual of A. For each $p \in \Gamma$ we define a σ-weakly continuous linear map ε_p, on M, by

$$\varepsilon_p(x) = \int_A \alpha_t(x)\overline{\langle t, p\rangle}\, dt \; (x \in M),$$

where dt is the normalized Haar measure on A. Let M_p be $\varepsilon_p(M)$. Then it is clear that $M_p = \{x \in M : \alpha_t(x) = \langle t, p\rangle x, t \in A\}$ and M_0 is the fixed point algebra. For every $p \in \Gamma$ define the projection

$$f_p = \sup\{uu^* : u \text{ is a partial isometry in } M_p\}.$$

Then $f_{-p} = \sup\{u^*u : u \text{ is a partial isometry in } M_p\}$ as $M_{-p} = M_p^*$. The following result is well known (see **[17]**).

LEMMA 2.1. *For every $p, q \in \Gamma$,*

(1) $f_p \in Z(M_0)$ *(the center of* M_0*)*;
(2) $M_pM_q \subseteq M_{p+q}$ *and* $M_p^* = M_{-p}$;
(3) *if* $x \in M_p$ *and* $x = v\,|x|$ *is its polar decomposition, then* $v \in M_p$ *and* $|x| \in M_0$.

We will need the following result.

LEMMA 2.2. *For every $p \in \Gamma$ there is a sequence of partial isometries $\{v_{p,n}\}_{n=0}^{\infty}$ with the following properties.*

(1) $v_{p,n}^*v_{p,m} = 0$ *if* $m \neq n$;
(2) $\sum_{m=0}^{\infty} v_{p,m}v_{p,m}^* = f_p$;
(3) *for each* $m \geq 1$, $v_{p,m}^*v_{p,m} \leq v_{p,m-1}^*v_{p,m-1}$;
(4) $M_p = \sum_{m=0}^{\infty} v_{p,m}M_0$ *(i.e. each $x \in M_p$ can be written as $\sum_{m=0}^{\infty} v_{p,m}x_m$, where $x_m \in M_0$ and the sum converges in the σ-weak operator topology)*;
(5) $M_p = M_0v_{p,0}M_0$ *(i.e. M_p is the σ-weak closure of the subspace spanned by $\{Av_{p,0}B : A, B \in M_0\}$).*

Proof. The existence of the partial isometries $\{v_{p,n}\}_{n=0}^{\infty}$ satisfying (1)–(4), was proved in [**16**, Proposition 2.3 and Theorem 2.4] for the case $\Gamma = \mathbf{Z}$. The proof in the general case is almost identical. For (5) simply note that for $m \geq 1$, $v_{p,m}^*v_{p,m} \leq v_{p,0}^*v_{p,0}$ and therefore $v_{p,m} = v_{p,m}v_{p,0}^*v_{p,0}v_{p,m}^*v_{p,m} \in M_0v_{p,0}M_0$. ∎

With the partial isometries $\{v_{p,m} : m \geq 0, p \in \Gamma\}$ defined as above we can define maps $\{\beta_p\}_{p\in\Gamma}$ on M_0' by the formula

$$\beta_p(T) = \sum_{m=0}^{\infty} v_{p,m}Tv_{p,m}^*.$$

We have the following results ([**17**, Lemma 2.4]).

LEMMA 2.3.

(1) *β_p is a well defined homomorphism from M_0' onto f_pM_0' that maps $Z(M_0)$ onto $f_pZ(M_0)$.*

(2) *β_p, restricted to $f_{-p}M_0'$ is a *-isomorphism of $f_{-p}M_0'$ onto f_pM_0' that maps $f_{-p}Z(M_0)$ onto $f_pZ(M_0)$;*

(3) *$\beta_p\beta_q(T)=\beta_{p+q}(f_{-q}T)=f_q\beta_{p+q}(T)$;*

(4) *$\beta_p(f_q)=\beta_p(\beta_q(I))=f_p\beta_{p+q}(I)=f_pf_{p+q}$.*

Since M_0 is σ-finite there is a faithful normal state w on M_0. Define w on M by

$$w(x)=w(\varepsilon_0(x)) \qquad (x \in M).$$

Then w is a faithful normal state on M_0 such that

$$w \circ \varepsilon_0 = w \quad \text{and} \quad w \circ \alpha_t = w \qquad (t \in A).$$

Considering the Gelfand–Naimark–Segal construction for w we may assume that M has a separating and cyclic vector $\rho_0 \in H$ such that

$$w(x)=\langle x\rho_0, \rho_0\rangle = \langle \varepsilon_0(x)\rho_0, \rho_0\rangle \qquad (x \in M).$$

As $Z(M_0)$ is an abelian von Neumann algebra on a separable Hilbert space H, there is a locally compact complete separable metric measure space (X, μ) such that H is (unitarily equivalent to) the direct integral of Hilbert spaces $\{H(x)\}$ over (X, μ) and $Z(M_0)$ is (unitarily equivalent to) the algebra of diagonalizable operators relative to this decomposition [**5**, Theorem 14.2.1]. Also, $Z(M_0)'$ is the algebra of decomposable operators.

For every $p \in \Gamma$, β_p defines a *-isomorphism from $f_{-p}Z(M_0)$ onto $f_pZ(M_0)$. There are subsets $\{\hat{f}_p : p \in \Gamma\}$ of X such that $x \to \chi_{\hat{f}_p}(x)$ is the decomposition of f_p. (Here χ_B is the characteristic function of $B \subseteq X$.) Then β_p induces a *-isomorphism, denoted also by β_p, from $L^\infty(\hat{f}_{-p}, \mu \mid \hat{f}_{-p})$ onto $L^\infty(\hat{f}_p, \mu \mid \hat{f}_p)$. Therefore there is an invertible Borel transformation $\hat{\beta}_p$ from $\hat{f}_{-p}$ onto $\hat{f}_p$ such that, for $g \in L^\infty(f_{-p}, \mu \mid \hat{f}_p)$, we have

$$\beta_p(g)=g \circ \hat{\beta}_p^{-1}$$

and

$$(\mu \mid \hat{f}_p) \circ \hat{\beta}_p \sim \mu \mid \hat{f}_{-p} \qquad (p \in \Gamma).$$

We now define a groupoid G as follows.

$$G=\{(x,p) : x \in \hat{f}_p, p \in \Gamma\},$$

$(x,p)(y,q)=(x,p+q)$ if $y=\hat{\beta}_p^{-1}(x)$ (and undefined otherwise) and $(x,p)^{-1}=(\hat{\beta}_p^{-1}(x), -p)$.

Using Lemma 2.3 it is easy to check that G is indeed a groupoid with this multiplication and inverse operation. (For definitions see [**3**].) We can make it a measured groupoid by defining the measure ν on G by:

$$\int_G f\,d\nu = \int_X \Big(\sum_{p\in\Gamma} f(\hat{\beta}_p(x), p)\Big)\,d\mu$$

for a Borel function f on G.

We will denote by R the principal groupoid associated with G; i.e. $R = \{(x, y) : y = \hat{\beta}_p^{-1}(x),\ p \in \Gamma,\ x \in \hat{f}_p\}$. Thus R is a measured equivalence relation (see [**2**]).

Let $\tilde{N}$ be $M \cap Z(M_0)'$. Then, for $t \in A$, $\alpha_t(\tilde{N}) = \tilde{N}$. Hence Lemma 2.1 can be applied to the algebra $\tilde{N}$ (in place of M) to get projections $Q_p \in Z(N_0)$ (where $N_0 = \tilde{N} \cap M_0 = \tilde{N} \cap Z(M_0)')$ such that

$$Q_p = \sup\{uu^* : u \text{ is a partial isometry in } N_p = \tilde{N} \cap M_p\}.$$

In fact we have the following result.

Lemma 2.4. *Let $\{Q_p\}$ be as above.*

(1) *Q_p is the largest subprojection of f_{-p}, in $Z(M_0)$, such that for every $Q \leq Q_p$, $Q \in Z(M_0)$ we have $\beta_p(Q) = Q$.*
(2) *For every non-zero projection $F \leq 1 - Q_p$, $F \in Z(M_0)$, there is a non zero projection $F' \leq F$ in $Z(M_0)$ such that $\beta_p(F')F' = 0$.*
(3) *$Q_p = Q_{-p} \leq f_p f_{-p}$.*
(4) *$Q_p Q_q \leq Q_{p+q}$.*
(5) *For $T \in M_p$, $Q_p T = T Q_p$.*
(6) *$Q_p M_p = M_p \cap Z(M_0)' (= N_p)$.*

Proof. By applying Lemma 2.2(2) to $\tilde{N}$ we can write Q_p as a sum $\sum_{m=0}^{\infty} u_m u_m^*$, where $u_m \in M_p \cap Z(M_0)'$. Now, for a partial isometry $w \in Q_p M_p$ we have $w = ww^*w = \sum u_m u_m^* w \in Z(M_0)'$ (as $u_m \in Z(M_0)'$ and $u_m^* w \in M_0$). Hence $w \in N_p$ and, since M_p is generated by partial isometries, $Q_p M_p \leq N_p$. Since $N_p \leq Q_p M_p$ by the way Q_p was defined, $N_p = Q_p M_p$. We have $Q_p M_p Q_p = Q_p M_p = N_p = (N_{-p})^* = (Q_{-p} M_{-p})^* = M_p Q_{-p}$. Hence $M_p Q_{-p}(1 - Q_p) = 0$ and, thus, $Q_{-p}(1 - Q_p) = f_{-p} Q_{-p}(1 - Q_p) = 0$. By symmetry $Q_p = Q_{-p} \leq f_p f_{-p}$. Therefore, $Q_p M_p = M_p Q_p$ for $p \in \Gamma$. Hence $\beta_p(Q_p) \leq Q_p$. By applying β_{-p} we get $Q_p \leq \beta_{-p}(Q_p)$ and, since this holds for all $p \in \Gamma$, $Q_p = Q_{-p} \leq \beta_p(Q_{-p}) = \beta_p(Q_p)$. Hence $Q_p = \beta_p(Q_p)$. For $Q \in Z(M_0)$, $Q \leq Q_p$ we have, $\beta_p(Q) = \sum v_{p,m} Q v_{p,m}^* = \sum (v_{p,m} Q_p) Q v_{p,m}^* = Q \sum v_{p,m} Q_p v_{p,m}^* = Q \beta_p(Q_p) = Q Q_p = Q$.

To complete the proof of part (1) suppose that Q' is a projection in $Z(M_0)$ such that $\beta_p(Q) = Q$ for every $Q \leq Q'$. Then, for every Q in $Z(M_0)$ and $m \geq 0$, we have

$$u_{p,m} Q' Q u_{p,m}^* = u_{p,m} u_{p,m}^* \beta_p(QQ') = u_{p,m} u_{p,m}^* QQ'.$$

Hence

$$u_{p,m} Q' Q = u_{p,m} u_{p,m}^* QQ' u_{p,m} = QQ' u_{p,m}.$$

Thus $Q' u_{p,m} \in Z(M_0)'$ for every $m \geq 0$. But then $Q' M_p \subseteq N_p$ and $Q' \leq Q_p$. This completes the proof of (1). To prove (2), fix a non-zero projection $F \leq 1 - Q_p$ in $Z(M_0)$. If

$F(1-f_{-p})\neq 0$ then, letting $F'=F(1-f_{-p})$ we have $\beta_p(F')=0$ and we are done. So assume $F\leq f_{-p}$. By the maximality property (1) of Q_p there is a projection $F''\leq F$ such that $\beta_p(F'')\neq F''$. Now either $F'=F''-\beta_p(F'')F''$ or $F'=\beta_p^{-1}(\beta_p(F'')-\beta_p(F'')F'')$ will do. This proves (2).

(3) was already proved above and (4) follows from $N_pN_q\subseteq N_{p+q}$. For (5), let T be in M_p; then $TQ_p\in Z(M_0)'$ and thus $TQ_p=Q_pTQ_p$. Also $T^*Q_p=Q_pT^*Q_p$, since $Q_p=Q_{-p}$. Hence $TQ_p=Q_pTQ_p=(Q_pT^*Q_p)^*=(T^*Q_p)^*=Q_pT$. ■

LEMMA 2.5. *Assume that* $Z(M)\cap M_0=CI$.

(1) *For every* $p,q\in\Gamma$, $f_{-p}\beta_q(Q_p)\leq Q_p$.

(2) *For every* $p\in\Gamma$ *either* $Q_p=I$ *or* $Q_p=0$.

(3) $N=\{p\in\Gamma:Q_p=I\}$ *is a subgroup of* Γ.

Proof. Let Q be a subprojection, in $Z(M_0)$, of $f_{-p}\beta_q(Q_p)$. Then

$$Q=Q\beta_q(Q_p)=f_qQ\beta_q(Q_p)=\beta_q(\beta_{-q}(Q))\beta_q(Q_p)=\beta_q(\beta_{-q}(Q)Q_p).$$

Write $F=\beta_{-q}(Q)Q_p$; then $F\leq Q_p\leq f_{-p}$ and $Q=\beta_p(F)$. We have $\beta_p(Q)=\beta_p(\beta_q(F))=\beta_p\beta_q(f_{-q}F)=\beta_{p+q}(f_{-q}F)=\beta_{p+q}(f_{-q}f_{-p}F)=\beta_q(\beta_p(f_{-q}F))=\beta_q(f_{-p}F)$, as $f_{-q}F\leq Q_p$. Hence $\beta_p(Q)=\beta_q(f_{-p}F)=\beta_q(F)=Q$. Since Q is arbitrary in $Z(M_0)$, $f_{-p}\beta_q(Q_p)\leq Q_p$. This proves (1). To prove (2) first note that, for every $p\in\Gamma$,

$$\bigvee_{q\in\Gamma}\beta_q(Q_p)\in Z(M_0)\cap Z(M)=CI.$$

Hence if $Q_p\neq 0$, $\bigvee_q\beta_q(Q_p)=I$ and, from (1), $f_{-p}\leq Q_p$. Hence if $Q_p\neq 0$, then $Q_p=f_p=f_{-p}$. Now write $F=1-f_p$. Then, for $q\in\Gamma$, we have

$$\begin{aligned}f_p\beta_q(F)&=Q_p\beta_q(F)=f_q\beta_p(Q_p\beta_p(F))=f_q\beta_p(\beta_q(F))\\&=f_q\beta_{p+q}(F)=\beta_q(\beta_p(F))=0.\end{aligned}$$

Hence if $F\neq 0$, then $\bigvee_q\beta_q(F)=I$ and $f_p=0$. Therefore if $Q_p\neq 0$, then $F=0$; i.e. $Q_p=f_p=I$.

Part (3) follows from the fact that, for $p,q\in\Gamma$, $Q_p=Q_{-p}$ and $Q_{p+q}\geq Q_pQ_q$. ■

Combining Lemma 2.5 with Lemma 2.4 we see that $M\cap Z(M_0)'$ is generated by $\cup\{M_p:p\in N\}$ where $N=\{p\in\Gamma:Q_p=I\}$.

As was mentioned above, we assume that there is a separating and cyclic vector $\rho_0\in H$ and that

$$\langle\alpha_t(x)\rho_0,\rho_0\rangle=\langle x\rho_0,\rho_0\rangle\qquad(x\in M,\ t\in A).$$

It follows that for $t\in A$, $W_tx\rho_0=\alpha_t(x)\rho_0$ defines a unitary operator on H and $t\to W_t$ is a homomorphism, continuous in the strong topology. Also note that $W_txW_t^*=\alpha_t(x)$ for $x\in M$.

Let $W_t=\sum_{p\in\Gamma}\langle t,p\rangle E_p$ be its spectral decomposition. Then it is easy to check that

$$E_px\rho_0=\varepsilon_p(x)\rho_0,\qquad p\in\Gamma,\qquad x\in M.$$

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Now, let N be as above and write, for $\gamma \in \Gamma/N$,

$$F_\gamma = \sum_{p+N=\gamma} E_p.$$

Then $\{F_\gamma\}_{\gamma\in\Gamma/N}$ is an orthogonal family of projections with sum I. Let

$$U_s = \sum_{\gamma\in\Gamma/N} \langle s, \gamma\rangle F_\gamma, \qquad s \in (\Gamma/N)\hat{}.$$

For $s \in \Gamma/N$, $p,q \in \Gamma$ we have

$$U_s x U_s^* y\rho_0 = U_s x \overline{\langle s, \pi(q)\rangle} y\rho_0 = \overline{\langle s, \pi(q)\rangle}\langle s, \pi(p+q)\rangle xy\rho_0 = \langle s, \pi(p)\rangle xy\rho_0$$

for $x \in M_p$, $y \in M_q$ (where π is the quotient map $\Gamma \to \Gamma/N$). Thus $U_s x U_s^* = \langle s, \pi(\rho)\rangle x$ for every $s \in (\Gamma/N)\hat{}$, $p \in \Gamma$. In particular $U_s M_p U_s^* \subseteq M$, $s \in (\Gamma/N)\hat{}$, $p \in \Gamma$. Hence $U_s M U_s^* \subseteq M$ and we write

$$\delta_s(x) = U_s x U_s^* \ (s \in (\Gamma/N)\hat{},\ x \in M).$$

This defines a σ-weakly continuous homomorphism δ of $(\Gamma/N)\hat{}$ into $\mathrm{Aut}(M)$. Also $U_s x\rho_0 = U_s x U_s^* \rho_0 = \delta_s(x)\rho_0$ and, if we write $\phi_\gamma(x) = \int_{(\Gamma/N)\hat{}} \langle \gamma, s\rangle \delta_s(x)\, ds$ (where ds is the Haar measure), we get $F_\gamma x\rho_0 = \phi_\gamma(x)\rho_0$ ($x \in M$, $\gamma \in \Gamma/N$).

The image of ϕ_0 is the fixed point algebra of δ; i.e. $\phi_0(M) = M^\delta = M \cap \{U_s\}'$. Hence ϕ_0 is an expectation onto $M \cap Z(M_0)'$.

LEMMA 2.6.

(1) *For* $p \in \Gamma$, $\gamma \in \Gamma/N$,

$$\phi_\gamma \circ \varepsilon_p = \varepsilon_p \circ \phi_\gamma = \begin{cases} 0 & \text{if } \pi(p) \neq \gamma \\ \varepsilon_p & \text{if } \pi(p) = \gamma \end{cases}.$$

(2) $\phi_{\pi(p)}(M)$ *is spanned, as a σ-weakly closed subspace, by* $\cup\{M_l : l \in p + N\}$.

Proof. For (1) simply observe that $(\phi_\gamma \circ \varepsilon_p)(x)\rho_0 = F_\gamma E_p x\rho_0 = E_p x\rho_0$ if $\pi(p) = \gamma$ and is 0 otherwise. For (2) note that M is spanned, as a σ-weakly closed subspace, by $\cup\{M_q : q \in \Gamma\}$; thus $\phi_{\pi(p)}(M)$ is spanned by $\cup\{\phi_{\pi(p)}(M_q) : q \in \Gamma\} = \cup\{M_q : q \in p + N\}$. ∎

LEMMA 2.7. *Fix $p \in \Gamma$ and a partial isometry $V \in \phi_\gamma(M)$, where $\gamma = \pi(p)$. Then, for every projection $F \in Z(M_0)$, we have*

$$VFV^* = VV^*\beta_p(F).$$

Proof. Note that $\beta_p(F)$ is the projection onto $[M_pF(H)]$ and for every $q \in p + N$, $\beta_q(F) = \beta_p(F)$; hence $\beta_p(F) = \bigvee_{q\in p+N} [M_qF(H)] = [\phi_{\pi(p)}(M)F(H)]$, since $\phi_{\pi(p)}(M)$ is spanned by $\cup\{M_q : q \in p + N\}$. As $[VF(H)] \subseteq [\phi_{\pi(p)}(M)F(H)] = [\beta_p(F)(H)]$, $VFV^* \leqslant VV^*\beta_p(F)$. Also $[VV^*M_pF(H)] = [VFV^*M_p(H)]$ (as $F \in Z(M_0)$ and $V^*M_p \in M \cap Z(M_0)'$). Hence $VV^*\beta_p(F) = VFV^*$. ∎

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3. The "matrix" representation. We will assume throughout the rest of the paper that $Z(M) \cap M_0 = \mathbf{C}I$.

Recall that (X, μ) is a locally compact complete separable metric space such that H is the direct integral of Hilbert spaces $\{H(x)\}$ over (X, μ) and $Z(M_0)$ is the algebra of diagonalizable operators relative to this decomposition.

For every $(x, y) \in R$ (the measured equivalence relation defined above) there is some $p \in \Gamma$ such that $y = \hat{\beta}_p^{-1}(x)$. We have $p + N \equiv \{p + q : q \in N\} = \{l \in \Gamma : \hat{\beta}_l^{-1}(x) = y\}$. Hence this defines a Borel map $d : R \to \Gamma/N$ that is a 1-cocycle; i.e. for almost every $(x, y, z) \in R^{(2)}$,

$$d(x, y) + d(y, z) = d(x, z).$$

(See [**2**] for cocycles on an equivalence relation.)

Lemma 3.1. *Fix $p \in \Gamma$ and a partial isometry $V \in \phi_\gamma(M)$, $\gamma = \pi(p)$. Then, for almost every $x \in X$, there is a partial isometry $\tilde{V}(x, \hat{\beta}_p(x))$ from $H(x)$ into $H(\hat{\beta}_p(x))$ such that*

$$(V\xi)(\hat{\beta}_p(x)) = D(x, \hat{\beta}_p(x))\tilde{V}(x, \hat{\beta}_p(x))\xi(x),$$

where $D(x, \hat{\beta}_p(x)) = \sqrt{\dfrac{d\mu}{d\mu \circ \beta_p}}(x)$.

Proof. Let $\{\xi_i\}$ a countable set in H that spans H. Fix $\xi \in H$ and a projection F in $Z(M_0)$. We have

$$\int_{\hat{F}} \|D^{-1}(x, \hat{\beta}_p(x))(V\xi)(\hat{\beta}_p(x))\|^2\, d\mu(x) = \int_{\hat{F}} \|(V\xi)(\hat{\beta}_p(x))\|^2\, d(\mu \circ \beta_p)(x)$$

$$= \int_{\hat{\beta}_p(\hat{F})} \|(V\xi)(y)\|^2\, d\mu(y) = \|\beta_p(F)V\xi\|^2 = \|VFV^*V\xi\|^2$$

$$= \|FV^*V\xi\|^2 = \int_{\hat{F}} \|(V^*V)(x)\xi(x)\|^2\, d\mu(x).$$

Since this holds for every Borel subset $\hat{F} \subseteq X$, $\|D^{-1}(x, \hat{\beta}_p(x))(V\xi)(\hat{\beta}_p(x)\| = \|(V^*V)(x)\xi(x)\|$ a.e. on X.

(Here $V^*V = \int V^*V(x)d\mu(x)$ is the decomposition of V^*V.) For every i there is a null set $N_i \subseteq X$ such that

$$\|D^{-1}(x, \hat{\beta}_p(x))(V\xi_i)(\hat{\beta}_p(x))\| = \|(V^*V)(x)\xi_i(x)\|$$

for every $x \notin N_i$. Let $N' = \cup N_i$. The above holds for every $x \notin N'$ and every i. Since $\{\xi_i(x)\}$ spans $H(x)$ for almost every x, the map

$$\xi_i(x) \to D^{-1}(x, \hat{\beta}_p(x))(V\xi_i)(\hat{\beta}_p(x))$$

can be extended to a partial isometry $\tilde{V}(x, \beta_p(x))$ from $H(x)$ (with initial projection $(V^*V)(x)$) into $H(\beta_p(x))$ (with final projection $VV^*(\beta_p(x))$). ∎

For an arbitrary $T \in \phi_\gamma(M)$ ($\gamma = \pi(p)$) let $T = V\,|T|$ be its polar decomposition and let

$$T(x, \hat{\beta}_p(x)) = D(x, \hat{\beta}_p(x))\tilde{V}(x, \hat{\beta}_p(x))\,|T|\,(x),$$

where $|T| = \int^\oplus |T|\,(x)\,d\mu(x)$ (as $T^*T \in \phi_0(M) = M \cap Z(M_0)'$) and D and $\tilde{V}$ are as in the last lemma. Then for a.e.x in X $T(x, \hat{\beta}_p(x))$ is a bounded operator from $H(x)$ into $H(\beta_p(x))$ such that for $\xi \in H$,

$$\begin{aligned} T(x, \hat{\beta}_p(x))\xi(x) &= D(x, \hat{\beta}_p(x))\tilde{V}(x, \tilde{\beta}_p(x))\,|T|\,(x)\xi(x) \\ &= D(x, \hat{\beta}_p(x))\tilde{V}(x, \hat{\beta}_p(x))(|T|\,\xi)(x) = (V\,|T|\,\xi)(\hat{\beta}_p(x)) = (T\xi)(\hat{\beta}_p(x)) \end{aligned}$$

for almost every x.

Clearly $T(x, \hat{\beta}_p(x)) = T(x, \hat{\beta}_q(x))$ if $p - q \in N$; so that we get a "matrix" representation of $T \in \phi_\gamma(M)$ over R. For an arbitrary $T \in M$ we define

$$T(x, y) = \phi_\gamma(T)(x, y), \quad \text{where } \gamma = d(x, y).$$

For $T \in \phi_\gamma(M)$ we have $\|T(x, y)\| \leq \|T\|\,D(x, y)$ and for $T \in M$,

$$\|T(x, y)\| \leq \|\phi_\gamma(T)\|\,D(x, y) \leq \|T\|\,D(x, y).$$

LEMMA 3.2. *Let $\mathcal{U} \subseteq M$ be an M_0-bimodule. Then for $T \in M$, $T \in \mathcal{U}$ if and only if $\phi_\gamma(T) \in \mathcal{U}$, for all $\gamma \in \Gamma/N$.*

Proof. Assume $T \in \mathcal{U}$. Let V be a partial isometry in $\phi_\gamma(M)$ satisfying $VZ(M_0)V^* \subseteq M_0$. Since $\phi_0(M) = M \cap Z(M_0)'$ and $Z(M_0)$ is an abelian von Neumann algebra, the results of [**1**, Theorem 6.2.2] show that $\phi_0(V^*T)$ lies in the σ-weakly closed convex hull of $\{UV^*TU^* : U \text{ is a unitary operator in } Z(M_0)\}$. Hence $V\phi_0(V^*T)$ lies in the σ-weakly closed convex hull of $\{(VUV^*)TU^* : U \text{ is a unitary operator in } Z(M_0)\}$ (since $VZ(M_0)V^* \subseteq M_0$).

Since $\phi_\gamma(M)$ is generated by $\{\varepsilon_p(M) : p \in \pi^{-1}(\gamma)\}$ we have $\phi_\gamma(M) = \left(\bigvee_{p \in \pi^{-1}(\gamma)} f_p\right)\phi_\gamma(M)$ and we can find a countable set of partial isometries $\{V_k\} \subseteq \phi_\gamma(M)$ such that $\sum V_kV_k^* = V\{f_p : p \in \pi^{-1}(\gamma)\}$ and $V_k \in M_p$ for some $p \in \pi^{-1}(\gamma)$. (See Lemma 2.2(2).)

For each such V_k we have $V_k\phi_0(V_k^*T) \in \mathcal{U}$. Also note that $V_k\phi_0(V_k^*T) = V_kV_k^*\phi_\gamma(T)$ (since it holds for every $T \in U\{\phi_\lambda(M) : \lambda \in \Gamma/N\}$ and ϕ_0, ϕ_γ are σ-weakly continuous). Hence $\phi_\gamma(T) = \sum V_kV_k^*\phi_\gamma(T) = \sum V_k\phi_0(V_k^*T) \in \mathcal{U}$. Since T is a σ-weak limit of finite linear combinations of $\{\phi_\gamma(T) : \gamma \in \Gamma/N\}$ (using an approximate identity on $(\Gamma/N)\hat{}$), it follows that T lies in $\mathcal{U}$. ■

LEMMA 3.3. *Let F and G be projections in M_0' and write $F = \int_X^\oplus F(x)\,d\mu(x)$, $G = \int_X^\oplus G(x)\,d\mu(x)$. Let $\mathcal{U}(F, G) = \{T \in M : (I - G)TF = 0\}$. Then $\mathcal{U}(F, G)$ is an M_0-bimodule and $\mathcal{U}(F, G) = \{T \in M : (1 - G(y))T(x, y)F(x) = 0 \text{ for almost all } (x, y) \in R\}$.*

Proof. $\mathcal{U}(F, G)$ is clearly a σ-weakly closed M_0-bimodule. Fix $\xi \in H$; $\xi = \int^{\oplus} \xi(x)\, d\mu(x)$. Then for $T \in M$, $\gamma \in \Gamma/N$ and $p \in \pi^{-1}(\gamma)$,

$$((1-G)\phi_\gamma(T)F\xi)(\hat{\beta}_p(x)) = (1-G(\hat{\beta}_p(x)))(\phi_\gamma(T)F\xi)(\hat{\beta}_p(x))$$
$$= (1-G(\hat{\beta}_p(x)))T(x, \hat{\beta}_p(x))F(x)\xi(x).$$

As ξ runs over a countable-set $\{\xi_i\}$ that spans H, $\{\xi_i(x)\}$ would span $H(x)$ and the equality above would hold for almost every $x \in X$. Hence $\phi_\gamma(T) \in \mathcal{U}(F, G)$ for all $\gamma \in \Gamma/N$ if and only if $(1-G(y))T(x, y)F(x) = 0$ for almost every $(x, y) \in R$. Lemma 3.2, applied to $\mathcal{U}(F, G)$, completes the proof. ∎

THEOREM 3.4. *Let $\mathcal{U}$ be a σ-weakly closed M_0-bimodule of M. Then we can find σ-weakly closed subspaces $\mathcal{U}(x, y)$, $(x, y) \in R$, of $M(x, y)$ such that $M_0(y)\mathcal{U}(x, y)M_0(x) \subseteq \mathcal{U}(x, y)$ for almost every $(x, y) \in R$ and*

$$\mathcal{U} = \{T \in M : T(x, y) \in \mathcal{U}(x, y) \text{ for almost every } (x, y) \in R\}.$$

Proof. Since M has a separating vector, all σ-weakly closed, linear subspaces of M are reflexive by Theorem 2.3 of **[9]**. Hence

$$\mathcal{U} = \{T \in M : T\xi \in [\mathcal{U}\xi] \text{ for all } \xi \in H\}.$$

Since $\mathcal{U}$ is an M_0-bimodule, the projection onto $[\mathcal{U}\xi]$ commutes with M_0 and

$$\mathcal{U} = \{T \in M : T[M_0\xi] \subseteq [\mathcal{U}\xi] \text{ for all } \xi\}.$$

So if $F(\xi)$ and $G(\xi)$ are the projections onto $[\mathcal{U}\xi]$ and $[M_0\xi]$ respectively, then $F(\xi)$ and $G(\xi)$ are in M_0' and

$$\mathcal{U} = \bigcap \{\mathcal{U}(F(\xi), G(\xi)) : \xi \in H\}.$$

In fact

$$\mathcal{U} = \bigcap \{\mathcal{U}(F(\xi), G(\xi)) : \xi \in H_0\},$$

where H_0 is a dense countable set in H. Hence

$$\mathcal{U} = \{T \in M : (I - G(\xi)(y))T(x, y)F(\xi)(x) = 0$$

for $\xi \in H_0$ and almost every $(x, y) \in R\}$. Set

$$\mathcal{U}(x, y) = \{S \in M(x, y) : (1 - G(\xi)(y))SF(\xi)(x) = 0 \text{ for all } \xi \in H_0\}.$$

Then we have

$$\mathcal{U} = \{T \in M : T(x, y) \in \mathcal{U}(x, y) \text{ for almost every } (x, y) \in R\}.$$

It is easy to check that $M_0(y)\mathcal{U}(x, y)M_0(x) \subseteq \mathcal{U}(x, y)$. ∎

LEMMA 3.5. *Let H_i $(i = 1, 2)$ be a Hilbert space, $M_i \subseteq B(H_i)$ be a σ-finite factor, $\mathbf{v} : H_1 \to H_2$ be a partial isometry such that $\mathbf{v}M_2\mathbf{v}^* \subseteq M_1$ and $\mathbf{v}^*M_1\mathbf{v} \subseteq M_2$. Let $\mathcal{U} \subseteq M_2\mathbf{v}M_1$ be a σ-weakly closed subspace such that $M_2\mathcal{U}M_1 \subseteq \mathcal{U}$. Then either $\mathcal{U} = \{0\}$ or $\mathcal{U} = M_2\mathbf{v}M_1$.*

Proof. Let $\mathbf{u}$ be a maximal partial isometry in $\mathcal{U}$ such that $\mathbf{u}^*\mathbf{u} \leqslant \mathbf{v}^*\mathbf{v}$ and $\mathbf{u}\mathbf{u}^* \leqslant \mathbf{v}\mathbf{v}^*$.

Then

$$(\mathbf{vv}^* - \mathbf{uu}^*)\mathcal{U}(\mathbf{v}^*\mathbf{v} - \mathbf{u}^*\mathbf{u}) = 0.$$

Hence, since $\mathcal{U} = M_2\mathcal{U}M_1$,

$$M_2(\mathbf{vv}^* - \mathbf{uu}^*)M_2\mathcal{U}M_1(\mathbf{v}^*\mathbf{v} - \mathbf{u}^*\mathbf{u})M_1 = 0.$$

Since M_i $(i = 1, 2)$ is a factor this implies that either $\mathcal{U} = M_2\mathcal{U}M_1 = 0$ or at least one of the two projections, $\mathbf{v}^*\mathbf{v} - \mathbf{u}^*\mathbf{u}$ or $\mathbf{vv}^* - \mathbf{uu}^*$, is zero. Suppose $\mathbf{v}^*\mathbf{v} = \mathbf{u}^*\mathbf{u}$. Then $\mathbf{v} = \mathbf{vv}^*\mathbf{v} = \mathbf{vu}^*\mathbf{u} \in M_2\mathbf{u} \subseteq \mathcal{U}$; and $\mathcal{U} = M_2\mathbf{v}M_1$. Similarly, if $\mathbf{vv}^* = \mathbf{uu}^*$, then $\mathcal{U} = M_2\mathbf{v}M_1$. ∎

COROLLARY 3.6 *If $M \cap Z(M_0)' \subseteq M_0$, then for every σ-weakly closed M_0-bimodule $\mathcal{U}$ of M and almost every $(x, y) \in R$, either $\mathcal{U}(x, y) = 0$ or $\mathcal{U}(x, y) = M(x, y)$; i.e. there is a subset $Q \subseteq R$ such that*

$$\mathcal{U} = \{T \in M : T(x, y) = 0 \text{ if } (x, y) \notin Q\}.$$

In particular, this is the case if α is inner.

Proof. Since $M \cap Z(M_0)' \subseteq M_0$, $R = G$ and $M(x, y) = M_0(y)u_p(x, y)M_0(x)$, where $y = \hat{\beta}_p(x)$ and u_p satisfies $M_0u_pM_0 = M_p$. Now apply Lemma 3.5. ∎

COROLLARY 3.7. *If $M \cap Z(M_0)' \subseteq M_0$ and $\mathcal{U}$ is a σ-weakly closed M_0-bimodule of M, then there are projections $\{e_p\}_{p\in\Gamma}$ in $Z(M_0)$ such that $\mathcal{U}$ is the σ-weakly closed subspace spanned by $\cup\{e_pM_p : p \in \Gamma\}$.*

Suppose $M \cap Z(M_0)' \subseteq M_0$. Then we see that there is a bijective correspondence between the Borel subsets of R (modulu sets of measure zero) and the σ-weakly closed M_0-bimodules of M. Write

$$\mathcal{U}(Q) = \{T \in M : T(x, y) = 0 \text{ if } (x, y) \notin Q\}.$$

Then one can easily show that $\mathcal{U}(Q)$ is an algebra if and only if $Q \circ Q \subseteq Q$ (where $(x, y) \cdot (y, z) = (x, z)$ is the multiplication in R); $\mathcal{U}(Q)$ is self adjoint if and only if $Q = Q^{-1}$ (where $(x, y)^{-1} = (y, x)$); and $\mathcal{U}(Q_1) \subseteq \mathcal{U}(Q_2)$ if and only if $Q_1 \subseteq Q_2$.

For the case where M_0 is a Cartan subalgebra of M similar results were proved in **[13]**.

Recall that we assume $Z(M) \cap M_0 = \mathbf{C}I$. We have the following result.

LEMMA 3.8. *If $M \cap Z(M_0)' \subseteq M_0$, then for $T \in M$, $t \in A$, we have*

$$\alpha_t(T)(x, y) = \langle p, t\rangle T(x, y)$$

for almost every $(x, y) \in R$ $(= G)$ (where $y = \hat{\beta}_p(x)$).

Proof. $\alpha_t(T)(x, y) = \varepsilon_p(\alpha_t(T))(x, \hat{\beta}_p(x)) = \langle p, t\rangle \varepsilon_p(T)(x, \hat{\beta}_p(x)) = \langle p, t\rangle T(x, y)$. (Here $\varepsilon_p = \phi_p$). ∎

PROPOSITION 3.9. *α is inner if and only if $G = R$ and the map $c : R \to \Gamma$ defined by $c(x, y) = p$, where $\hat{\beta}_p(x) = y$, is a coboundary; i.e. there is a Borel map $g : X \to \Gamma$ such that $c(x, y) = g(y) - g(x)$ for almost every $(x, y) \in R$.*

Proof. If α is inner, then $G = R$ as $M \cap Z(M_0)' \subseteq M_0$; also we have then a group $t \to U_t$ of unitary operators in $Z(M_0)$ such that $\alpha_t(T) = U_t T U_t^*$ $(T \in M, t \in A)$. There is a function $g : X \to \Gamma$ such that for almost every $x \in X$, $\langle g(x), t \rangle = U_t(x)$ (identify U_t with a Borel function on X). For $T \in \varepsilon_p(M)$ we have, for almost every $x \in X$,

$$\langle p, t \rangle T(x, \hat{\beta}_p(x)) = \alpha_t(T)(x, \hat{\beta}_p(x)) = U_t(\hat{\beta}_p(x)) T(x, \hat{\beta}_p(x)) U_t^*(x)$$
$$= U_t(\hat{\beta}_p(x)) U_t^*(x) T(x, \hat{\beta}_p(x)).$$

Hence

$$\langle p, t \rangle = g(\hat{\beta}_p(x)) - g(x).$$

For the other direction, suppose such g exists and write $U_t(x) = \langle g(x), t \rangle$; then this defines a group of unitary operators in $Z(M_0)$ satisfying $\alpha_t(T) = U_t T U_t^*$. ■

4. M-Reflexivity. For a σ-weakly closed subspace $\mathcal{U}$ of M we let $\mathcal{L}(\mathcal{U}) = \{(P, Q) : P, Q$ are projections in M such that $P\mathcal{U}Q = 0\}$ and

$$\mathcal{S}\mathcal{L}(\mathcal{U}) = \{T \in M : PTQ = 0 \text{ for every } (P, Q) \in \mathcal{L}(\mathcal{U})\}.$$

We say that $\mathcal{U}$ is *M-reflexive* (see **[7]** and **[8]**) if

$$\mathcal{U} = \mathcal{S}\mathcal{L}(\mathcal{U}).$$

Now write

$$\mathcal{L}_0(\mathcal{U}) = \{(P, Q) : P, Q \text{ are projections in } M \cap M_0' \text{ such that } P\mathcal{U}Q = 0\}$$

and

$$\mathcal{S}\mathcal{L}_0(\mathcal{U}) = \{T \in M : PTQ = 0 \text{ for every } (P, Q) \in \mathcal{L}_0(\mathcal{U})\}.$$

LEMMA 4.1. *Let $\mathcal{U}$ be a σ-weakly closed M_0-bimodule in M. Then $\mathcal{U}$ is reflexive if and only if*

$$\mathcal{S}\mathcal{L}_0(\mathcal{U}) = \mathcal{U}.$$

Proof. For a projection $P \in M$ we write

$$R(P) = \sup\{UPU^* : U \in M_0 \text{ is a unitary operator}\}.$$

Then $R(P)$ is a projection in $M \cap M_0'$. If $P\mathcal{U}Q = 0$ (P, Q are projections in M) then for all unitary operators U, V in M_0, $UPU^*\mathcal{U}VQV^* \subseteq UP\mathcal{U}QV^* = 0$ (as $\mathcal{U}$ is an M_0-bimodule). Hence $(R(P), R(Q)) \in \mathcal{L}_0(\mathcal{U})$ whenever $(P, Q) \in \mathcal{L}(\mathcal{U})$. In fact $\mathcal{L}_0(\mathcal{U}) = \{(R(P), R(Q)) : (P, Q) \in \mathcal{L}(\mathcal{U})\}$. Also, if P, Q are projections in M and $(R(P), R(Q)) \in \mathcal{L}_0(\mathcal{U})$ then $(P, Q) \in \mathcal{L}(\mathcal{U})$ (as $P \leq R(P)$, $Q \leq R(Q)$).

If $T \in \mathcal{S}\mathcal{L}(\mathcal{U})$, then $T \in \mathcal{S}\mathcal{L}_0(\mathcal{U})$, since $\mathcal{L}_0(\mathcal{U}) \subseteq \mathcal{L}(\mathcal{U})$. If $T \in \mathcal{S}\mathcal{L}_0(\mathcal{U})$, then for every $(P, Q) \in \mathcal{L}(\mathcal{U})$, $(R(P), R(Q)) \in \mathcal{L}_0(\mathcal{U})$ and, therefore, $R(P)TR(Q) = 0$ $P \leq R(P)$, $Q \leq R(Q)$, $PTQ = 0$ and $T \in \mathcal{S}\mathcal{L}(\mathcal{U})$. Therefore $\mathcal{S}\mathcal{L}(\mathcal{U}) = \mathcal{S}\mathcal{L}_0(\mathcal{U})$.

LEMMA 4.2. *For $\gamma \in \Gamma/N$ and a projection $E \in Z(M_0)$, $E\phi_\gamma(M)$ is M-reflexive.*

Proof. For $\gamma \in \Gamma/N$ and a projection $E \in Z(M_0)$, we have

$$\mathcal{L}_0(E\phi_\gamma(M)) = \{(P, Q) : P, Q \in M \cap M_0',\ PE[\phi_\gamma(M)Q(H)] = 0\}.$$

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Since $[\phi_\gamma(M)Q(H)] = \beta_p(Q)(H)$ for every $p \in \pi^{-1}(\gamma)$,

$$\mathcal{L}_0(E\phi_\gamma(M)) = \{(P, Q) : P, Q \in M \cap M_0', \ EP\beta_p(Q) = 0\}$$

($p \in \pi^{-1}(\gamma)$ is now fixed). If $T \in \mathcal{S}\mathcal{L}_0(E\phi_\gamma(M))$, then $PTQ = 0$ whenever $EP\beta_p(Q) = 0$. Hence, for $\lambda \in \Gamma/N$, $P\phi_\lambda(T)Q = 0$ whenever $EP\beta_p(Q) = 0$. (Note that $\mathcal{S}\mathcal{L}_0(E\phi_\gamma(M))$ is a σ-weakly closed M_0-bimodule and, thus, $\phi_\lambda(\mathcal{S}\mathcal{L}_0(E\phi_\gamma(M))) \subseteq \mathcal{S}\mathcal{L}_0(E\phi_\gamma(M))$). Note that $\phi_\lambda(T)Q(H) \subseteq \beta_q(Q)(H)$ and $\phi_\lambda(T)(I - Q)(H) \subseteq \beta_q(1 - Q)(H)$ for every $q \in \pi^{-1}(\lambda)$. Fix such q. Then

$$\begin{aligned}\beta_q(Q)\phi_\lambda(T) &= \beta_q(Q)\phi_\lambda(T)(1 - Q) + \beta_q(Q)\phi_\lambda(T)Q \\ &= \beta_q(Q)\beta_q(I - Q)\phi_\lambda(T)(1 - Q) + \phi_\lambda(T)Q = \phi_\lambda(T)Q.\end{aligned}$$

Hence $P\beta_q(Q)\phi_\lambda(T) = 0$ whenever $P\beta_p(Q)E = 0$. Since $(1 - E)\beta_p(I)E = 0$ we have $(1 - E)f_q\phi_\lambda(T) = 0$; i.e. $(1 - E)\phi_\lambda(T) = 0$. Suppose $\gamma \neq \lambda$ and write

$$F = \sup\{Q(1 - \beta_{p-q}(Q)) : Q \text{ is a projection in } Z(M_0)\}.$$

Then, by Lemma 2.4(2), if $F \neq I$, there is a non zero projection $F' \leq 1 - F$ in $Z(M_0)$ such that $\beta_{p-q}(F')F' = 0$. But then $F' = F'(1 - \beta_{p-q}(F')) \leq F$. Hence $F = I$. Since for every $Q \in Z(M_0)$, $E(1 - \beta_p(Q))\beta_q(Q) = 0$ we have,

$$(1 - \beta_p(Q))\beta_q(Q)\phi_\lambda(T) = 0.$$

But $(1 - \beta_p(Q))\beta_q(Q) = \beta_q(Q(1 - \beta_{p-q}(Q)))$; hence

$$0 = \beta_q(I)\phi_\lambda(T) = f_q\phi_\lambda(T) = \phi_\lambda(T).$$

Therefore $T = E\phi_\gamma(T) \in E\phi_\gamma(M)$. ∎

COROLLARY 4.3. *Suppose $M \cap Z(M_0)' \subseteq M_0$ and γ is an automorphism of M with $\gamma(a) = a$ for $a \in Z(M_0)$. Then for every M_0-bimodule $\mathcal{U}$, $\gamma(\mathcal{U}) = \mathcal{U}$.*

Hence every von Neumann subalgebra $M \supseteq B \supseteq M_0$ is an image of a faithful normal expectation from M onto B.

Proof. Let γ be an automorphism as above and note that for every pair of projections P, Q in $Z(M_0)$ and $p \in \Gamma$, $P\gamma(M_p)Q = 0$ if and only if $PM_pQ = 0$. The M-reflexivity of M_p now implies that $\gamma(M_p) = M_p$. Corollary 3.7 shows that $\gamma(\mathcal{U}) = \mathcal{U}$ for every M_0-bimodule $\mathcal{U}$. The last statement of the corollary follows from Takesaki's Theorem **[18]** applied to $w(x) = \langle x\rho_0, \rho_0\rangle$ since $\sigma_t^w(a) = a$ for $a \in Z(M_0)$

$$\begin{aligned}(\text{as } w(ax) &= \langle ax\rho_0, \rho_0\rangle = \langle \varepsilon_0(ax)\rho_0, \rho_0\rangle = \langle a\varepsilon_0(x)\rho_0, \rho_0\rangle \\ &= \langle \varepsilon_0(x)a\rho_0, \rho_0\rangle = \langle xa\rho_0, \rho_0\rangle, \ a \in Z(M_0), \ x \in M).\end{aligned}$$ ∎

THEOREM 4.4. *The following statements are equivalent.*

(1) *α is inner.*

(2) *For every non-zero projection $F \in Z(M_0)$ there is a non-zero projection $Q \leq F$, $Q \in Z(M_0)$, such that for every $0 \neq p \in \Gamma$, we have $Q\beta_p(Q) = 0$.*

(3) *$M \cap Z(M_0)' \subseteq M_0$ and every σ-weakly closed M_0-bimodule is M-reflexive.*

Proof. The equivalence of (1) and (2) can be derived from [**4**, Theorem 1.1(iii)] or [**6**, Theorem 4.9]. One can also use Proposition 3.9 (applied to FMF instead of M) and the fact that a cocycle is a coboundary if and only if its only essential value is $\{0\}$. (See [**15**, Theorem 3.9(4)].) If α is inner, then clearly $M \cap Z(M_0)' \subseteq M_0$. We will now show that if α is inner and $\mathcal{U}$ is a σ-weakly closed M_0-bimodule then $\mathcal{U}$ is M-reflexive. Using Corollary 3.7 we can see that $\mathcal{U}$ is the σ-weakly closed subspace spanned by $U\{e_p\varepsilon_p(M) : p \in \Gamma\}$ (here $N = \{0\}$) for some projections $\{e_p\}_{p\in\Gamma}$ in $Z(M_0)$. For every projection Q in $Z(M_0)$ that satisfies $Q\beta_p(Q) = 0$ for every $0 \neq p \in \Gamma$ we have $\beta_q(Q)\beta_p(Q) = 0$ if $q \neq p$. Fix $q \in \Gamma$ and write $G = \beta_q(G)$. Then for every $p \neq q$, $Ge_p\beta_p(Q) = 0$ and, thus, $((1-e_q)G, Q) \in \mathcal{L}_0(\mathcal{U})$. If $T \in \mathcal{SL}_0(\mathcal{U})$, then

$$(1-e_q)\beta_q(Q)TQ = 0$$

for every $q \in \Gamma$. Hence for every $q \in \Gamma$, $(1-e_q)\beta_q(Q)\varepsilon_q(T)Q = 0$. Since $\beta_q(Q)\varepsilon_q(T)Q = \varepsilon_q(T)Q$, we have

$$(1-e_q)\varepsilon_q(T)Q = 0.$$

Now (2) implies that $I = \bigvee\{Q \in Z(M_0) : Q\beta_p(Q) = 0 \text{ for every } p \neq 0\}$. Therefore, $\varepsilon_q(T) \in e_q\varepsilon_q(M)$ and $T \in \mathcal{U}$.

We now turn to the proof that (3)$\Rightarrow$(2). For this fix $0 \neq F$ in $Z(M_0)$ and let $\mathcal{U} = \{T \in FMF : \varepsilon_0(T) = 0\}$. This is a σ-weakly closed M_0-bimodule and thus is M-reflexive (assuming statement (3)).

Since $M \cap M_0' \subseteq Z(M_0)$ (as $M \cap M_0' \subseteq M \cap Z(M_0)' \subseteq M_0$), we have

$$\begin{aligned}\mathcal{L}_0(\mathcal{U}) &= \{(P, L) : P, L \text{ are projections in } Z(M_0) \text{ such that } P\mathcal{U}L = 0\}\\ &= \{(P, L) : P, L \in Z(M_0),\ F\beta_p(F)P\beta_p(L) = 0 \text{ for every } p \neq 0\}.\end{aligned}$$

Since $F \notin \mathcal{SL}_0(\mathcal{U})$, there is some $(P, L) \in \mathcal{L}_0(\mathcal{U})$ such that $FPL \neq 0$. Write $Q = PFL$. Then for $p \neq 0$, $\beta_p(Q) \leq \beta_p(L) \leq \beta_p(F)(1-FP)$ and $Q\beta_p(Q) = 0$. This proves (2). ■

5. Isomorphisms. We assume now that M and B are factors, A_1 and A_2 are compact abelian groups, α and η are representations of A_1 and A_2, respectively, as *-automorphism groups on M and B respectively. Write $\Gamma_i = \hat{A}_i$ and define M_p ($p \in \Gamma_1$) and B_q ($q \in \Gamma_2$) as in Section 1.

We will assume that $M \cap Z(M_0)' = M_0$ and $B \cap Z(B_0)' = B_0$. Also let $\Sigma_i \subseteq \Gamma_i$ be a positive semigroup for $i = 1,2$; i.e. $\Sigma_i + \Sigma_i \subseteq \Sigma_i$ and $\Sigma_i \cap (-\Sigma_i) = \{0\}$. We write $M^\alpha(\Sigma_1)$ and $B^\eta(\Sigma_2)$ for the associated analytic subalgebras of M and B respectively; i.e. $M^\alpha(\Sigma_1)$ is the σ-weakly closed subspace spanned by $\bigcup\{M_p : p \in \Sigma_1\}$ and $B^\eta(\Sigma_2)$ is the σ-weakly closed subspace spanned by $\bigcup\{B_q : q \in \Sigma_2\}$.

Also, let $R_1 \subseteq X_1 \times X_1$ and $\{\beta_p^1 : p \in \Gamma_1\}$ be the equivalence relation and the maps associated with (M, α) and $R_2 \subseteq X_2 \times X_2$ and $\{\beta_q^2 : q \in \Gamma_2\}$ be the ones associated with (B, η). Let

$$P_1 = \{(x, y) \in R_1 : y = \hat{\beta}_p^1(x),\ p \in \Sigma_1\},$$

$$P_2 = \{(x, y) \in R_2 : y = \hat{\beta}_q^2(x),\ q \in \Sigma_2\}$$

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and note that

$$M^{\alpha}(\Sigma_1)=\{T\in M:\operatorname{supp}T\subseteq P_1\},$$

$$B^{\eta}(\Sigma_2)=\{T\in B:\operatorname{supp}T\subseteq P_2\},$$

where $\operatorname{supp} T=\{(x,y)\in R_i : T(x,y)\neq 0\}$ is defined up to a set of measure zero (and so is the inclusion $\operatorname{supp} T\subseteq P_i$ above).

If Σ_i totally orders Γ_i (i.e. $\Sigma_i\cup(-\Sigma_i)=\Gamma_i$), then $P_i\cup P_i^{-1}=R_i$ (up to a set of measure zero), where $(x,y)^{-1}=(y,x)$.

The main result of this section is the following theorem.

THEOREM 5.1. *Let $M^{\alpha}(\Sigma_1)$ and $B^{\eta}(\Sigma_2)$ be as above, and let ψ be an algebraic isomorphism from $M^{\alpha}(\Sigma_1)$ onto $B^{\eta}(\Sigma_2)$ such that, for $a\in M_0$, we have $\psi(a)^*=\psi(a^*)$. Then*

(1) *$\psi(M_0)=B_0$. (Write $\gamma:X_1\to X_2$ for the invertible Borel map that implements $\psi:Z(M_0)\to Z(B_0)$.)*

(2) *$B^{\eta}(\Sigma_2)$ is the σ-weakly closed subspace spanned by $\cup\{\psi(M_p):p\in\Sigma_1\}$.*

(3) *$\gamma\times\gamma(P_1)=P_2$ (where $(\gamma\times\gamma)(x,y)=(\gamma(x),\gamma(y))$) and, if Σ_i totally orders Γ_i, $i=1,2$, then $(\gamma\times\gamma)(R_1)=R_2$.*

When ψ is the identity map we get the following result.

COROLLARY 5.2. *If $M=B$, $M^{\gamma}(\Sigma_1)=M^{\eta}(\Sigma_2)$ and Σ_i totally orders Γ_i $(i=1,2)$, then $R_1=R_2$ and $P_1=P_2$ (although the maps $\{\hat{\beta}^1_p\}$ and $\{\hat{\beta}^2_q\}$ might be different). Hence the equivalence relation R and the partial order P associated with an analytic subalgebra (satisfying $M\cap Z(M_0)'=M_0$) is unique.*

REMARK. In special cases more can be said about an isomorphism ψ as in the theorem. For the case when M_0 and B_0 are Cartan subalgebras see **[13]** and for the case where $M^{\alpha}(\Sigma_1)$ and $B^{\eta}(\Sigma_2)$ are analytic crossed products with $\Gamma_i=Z$ and $\Sigma_i=Z_+$, see **[11]** and **[12]**.

For the proof of the theorem we need a few lemmas. In the discussion and lemmas that follow we assume that the hypothesis of the theorem holds.

LEMMA 5.3 $\psi(M_0)=B_0$.

Proof. For $a\in M_0$, a^* is in M_0. Hence $\psi(a^*)=\psi(a)^*$ lies in $B^{\eta}(\Sigma_2)\cap B^{\eta}(\Sigma_2)^*=B_0$, so that $\psi(M_0)\subseteq B_0$.

Now, if $T\in B_0$, then $T\in Z(B_0)'\subseteq\psi(Z(M_0))'$. Hence $\psi^{-1}(T)\in M\cap Z(M_0)'=M_0$. Hence $\psi(M_0)=B_0$. ■

Let $\overline{\psi(M_p)}$ be the σ-weak closure of $\psi(M_p)\subseteq B^{\eta}(\Sigma_2)$ for $p\in\Gamma$. It is a σ-weakly closed B_0-bimodule of B and, thus, there is a Borel set $C_p\subseteq P_2$ such that $\overline{\psi(M_p)}=\mathcal{U}(C_p)$, where $\mathcal{U}(Q)=\{T\in M:\operatorname{supp}T\subseteq Q\}$.

For an operator T we write $\operatorname{rp}(T)$ for the range projection of T. Using the definition of β_p in Section 2 one can see that for a projection $F\in Z(M_0)$, $\beta_p(F)=V\{\operatorname{rp}(TF):T\in M_p\}$.

LEMMA 5.4. *For $p \in \Gamma_1$ and a projection F in $Z(M_0)$ we have*

$$\psi(\beta_p^1(F)) = V\{\mathrm{rp}(S\psi(F)) : S \in \mathcal{U}(C_p)\} \in Z(B_0).$$

Proof. First note that

$$V\{\mathrm{rp}(S\psi(F)) : S \in \mathcal{U}(C_p)\} = V\{\mathrm{rp}(\psi(TF)) : T \in M_p\}$$

since $\mathcal{U}(C_p) = \overline{\psi(M_p)}$. Now, for $T \in M_p$ and $F \in Z(M_0)$ write $Q = \mathrm{rp}(TF)$. Let L be a projection in $Z(M_0)$. Then, by Lemma 2.7 and Lemma 2.1 (3), $LT = T\beta_{-p}(L)$ (write $T = |T|\, V^*$ and use $LV^* = V^*\beta_{-p}(L)$ as $V^* \in M_{-p}$). Hence, for a unitary operator U in $Z(M_0)$, we have

$$U^*T = T\beta_{-p}(U^*).$$

Thus,

$$\begin{aligned} UQU^*TF &= UQT\beta_{-p}(U^*)F = UQTF\beta_{-p}(U^*) = UTF\beta_{-p}(U^*) \\ &= UT\beta_{-p}(U^*)F = UU^*TF = TF, \end{aligned}$$

so that $UQU^* \geq Q$ for every unitary $U \in Z(M_0)$; hence $Q \in M \cap Z(M_0)' = M_0$.

Since ψ, restricted to M_0, is a *-isomorphism of M_0 onto B_0 (Lemma 5.3) and $\mathrm{rp}(TF) \in M_0$ for every $T \in M_p$, we have

$$\psi(\beta_p(F)) = \psi(V\{\mathrm{rp}(TF) : T \in M_p\}) = V\{\psi(\mathrm{rp}(TF)) : T \in M_p\}.$$

Notice that, for $T \in M_p$,

$$\psi(\mathrm{rp}(TF))\psi(TF) = \psi(\mathrm{rp}(TF)TF) = \psi(TF);$$

hence $\psi(\mathrm{rp}(TF)) \geq \mathrm{rp}(\psi(TF))$. Also

$$\psi^{-1}(\mathrm{rp}(\psi(TF))TF = \psi^{-1}(\mathrm{rp}(\psi(TF))\psi(TF)) = \psi^{-1}(\psi(TF)) = TF;$$

hence

$$\psi^{-1}(\mathrm{rp}(\psi(TF))\mathrm{rp}(TF) = \mathrm{rp}(TF) \text{ and } \psi(\mathrm{rp}(TF)) \leq \mathrm{rp}(\psi(TF)).$$

Therefore $\psi(\mathrm{rp}(TF)) = \mathrm{rp}(\psi(TF))$ and we have,

$$\psi(\beta_p)(F) = V\{\mathrm{rp}(\psi(TF) : T \in M_p\} = V\{\mathrm{rp}(S\psi(F)) : S \in \mathcal{U}(C_p)\}. \qquad \blacksquare$$

For a map $\phi : X_i \to X_i$ we write

$$g(\phi) = \{(x, \phi(x)) \in X_i \times X_i\}.$$

LEMMA 5.5. *Suppose $\hat{L}$ is a Borel subset of X_2 and $\lambda \in \Sigma_2$ satisfies*

$$g(\hat{\beta}_\lambda^2) \cap (X_2 \times \hat{L}) \subseteq C_p.$$

Then

$$g(\hat{\beta}_\lambda^2) \cap (X_2 \times \hat{L}) \subseteq g(\gamma \circ \hat{\beta}_p^1 \circ \gamma^{-1}),$$

where $\gamma : X_1 \to X_2$ implements ψ (viewed as an isomorphism of $Z(M_0) \simeq L^\infty(X_1, \mu_1)$ onto $Z(B_0) \simeq L^\infty(X_2, \mu_2)$).

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Proof. Let L be the projection in $Z(B_0)$ associated with $\hat{L}$. For $T \in B_\lambda$, TL is supported on $g(\beta_\lambda^2) \cap (X_2 \times \hat{L})$; hence on $C_p \cap (X_2 \times \hat{L})$. Thus $TL \in \mathcal{U}(C_p)L$. We have, using Lemma 5.4,

$$\beta_\lambda^2(\psi(F)L) = V\{\mathrm{rp}(T\psi(F)L : T \in B_\lambda\} \leq V\{\mathrm{rp}(S\psi(F)L) : S \in \mathcal{U}(C_p)\}$$
$$= \psi(\beta_p^1(F\psi^{-1}(L))) = \psi \circ \beta_p^1 \circ \psi^{-1}(\psi(F)L)$$

for every projection F in $Z(M_0)$. Thus

$$g(\beta_\lambda^2) \cap (X_2 \times \hat{L}) \subseteq g(\gamma \circ \beta_p^1 \circ \gamma^{-1}) \cap (X_2 \times \hat{L}). \qquad \blacksquare$$

LEMMA 5.6. $C_p = g(\gamma \circ \beta_p^1 \circ \gamma^{-1})$.

Proof. For $\lambda \in \Sigma_2$ let L_λ be the largest subprojection of $\psi(f_{-p})\beta_\lambda^{-1}(I)$ in $Z(B_0)$ such that $g(\hat{\beta}_\lambda^2) \cap (X_2 \times \hat{L}_\lambda) \subseteq C_p$, where $\hat{L}_\lambda$ is the associated Borel subset of X_2. For $\lambda_1 \neq \lambda_2$ let $L_0 = L_{\lambda_1}L_{\lambda_2}$; then, by Lemma 5.5,

$$g(\hat{\beta}_{\lambda_1}^2) \cap (X_2 \times \hat{L}_0) \subseteq g(\gamma \circ \hat{\beta}_p^1 \circ \gamma^{-1}).$$

and

$$g(\hat{\beta}_{\lambda_2}^2) \cap (X_2 \times \hat{L}_0) \subseteq g(\gamma \circ \hat{\beta}_p^1 \circ \gamma^{-1}).$$

But this implies that $L_0 = 0$, as $g(\hat{\beta}_{\lambda_1}^2) \cap g(\hat{\beta}_{\lambda_2}^2)$ is empty. Hence $g(\hat{\beta}_\lambda^2) \cap (X_2 \times \hat{L}_\lambda) = C_p \cap (X_2 \times \hat{L}_\lambda)$ for $\lambda \in \Sigma_2$. For $F \in Z(M_0)$ we now have,

$$\beta_\lambda^2(\psi(F)L) = V\{\mathrm{rp}(T\psi(F)L) : T \in B_\lambda\} = V\{\mathrm{rp}(S\psi(F)L : S \in \mathcal{U}(C_p)\}$$
$$= \psi \circ \beta_p^1 \circ \psi^{-1}(\psi(F)L).$$

Hence

$$C_p \cap (X_2 \times \hat{L}_\lambda) = g(\hat{\beta}_\lambda^2) \cap (X_2 \times \hat{L}_\lambda) = g(\gamma \circ \hat{\beta}_p^1 \circ \gamma^{-1}) \cap (X_2 \times \hat{L}_\lambda).$$

Since $V\{L_\lambda : \lambda \in \Sigma_2\} = \psi(f_{-p})$ and $\mathcal{U}(C_p)\psi(f_{-p}) = \mathcal{U}(C_p)$, $C_p = g(\gamma \circ \hat{\beta}_p^1 \circ \gamma^{-1})$. $\blacksquare$

LEMMA 5.7. *We have* $P_2 = \bigcup\{C_p : p \in \Sigma_1\} = \gamma \times \gamma(P_1)$ *and*

$$B^\eta(\Sigma_2) = \overline{\bigcup\{\psi(M_p) : p \in \Sigma_1\}},$$

where the closure is in the σ-weak topology.

Proof. Let $\mathcal{U} = \overline{\bigcup\{\psi(M_p) : p \in \Sigma_1\}}$. Since $\mathcal{U}$ is a σ-weakly closed B_0-bimodule of B, there is a set $C \subseteq P_2$ such that $\mathcal{U} = \mathcal{U}(C)$. Since $\psi(M_p) \subseteq \mathcal{U}$ for $p \in \Gamma_1$, $C_p \subseteq C$; hence $\mathcal{U}(\bigcup_p C_p) \subseteq \mathcal{U} = \mathcal{U}(C)$. But also $\mathcal{U} \subseteq \bigcup\{\mathcal{U}(C_p) : p \in \Sigma_1\} \subseteq \mathcal{U}(\bigcup_p C_p)$. Hence $\mathcal{U} = \mathcal{U}(\bigcup_p C_p)$. Write $Q = P_2 \setminus \bigcup_p C_p$ and assume $\nu_2(Q) > 0$. (ν_2 is a measure on R_2). Then there is some $0 \neq T \in B$ and $\lambda \in \Sigma_2$ such that $T \in B_\lambda$ and supp $T \subseteq Q$. Hence $\psi^{-1}(T) \neq 0$. Now Lemma 5.6, applied to ψ^{-1}, yields

$$\mathcal{U}(g(\gamma^{-1} \circ \hat{\beta}_\lambda^2 \circ \gamma)) = \psi^{-1}(B_\lambda).$$

Hence supp $\psi^{-1}(T) \subseteq g(\gamma^{-1} \circ \beta_\lambda^2 \circ \gamma)$. Therefore, there is some $q \in \Sigma_1$ and a projection $Z \in Z(M_0)$ such that

$$0 \neq Z\psi^{-1}(T) \in M_q.$$

Hence $0 \neq \psi(Z\psi^{-1}(T)) = \psi(Z)T$ and supp $\psi(Z)T \subseteq Q \cap C_q = \emptyset$. This contradiction shows that $P_2 = \bigcup\{C_p : p \in \Sigma_1\}$ and completes the proof of the lemma. ∎

To complete the proof of Theorem 5.1 just note that if $P_i \cup P_i^{-1} = R_i$ and $(\gamma \times \gamma)(P_1) = P_2$ then $(\gamma \times \gamma)(R_1) = R_2$. ∎

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