Canad. J. Math. Vol. 70 (2), 2018 pp. 426–450
http://dx.doi.org/10.4153/CJM-2017-002-7

Euler-type Relative Equilibria and their Stability in Spaces of Constant Curvature

Ernesto Pérez-Chavela and Juan Manuel Sánchez-Cerritos

Abstract. We consider three point positive masses moving on S^2 and H^2. An Eulerian-relative equilibrium is a relative equilibrium where the three masses are on the same geodesic. In this paper we analyze the spectral stability of these kind of orbits where the mass at the middle is arbitrary and the masses at the ends are equal and located at the same distance from the central mass. For the case of S^2, we found a positive measure set in the set of parameters where the relative equilibria are spectrally stable, and we give a complete classification of the spectral stability of these solutions, in the sense that, except on an algebraic curve in the space of parameters, we can determine if the corresponding relative equilibrium is spectrally stable or unstable. On H^2, in the elliptic case, we prove that generically all Eulerian-relative equilibria are unstable; in the particular degenerate case when the two equal masses are negligible, we get that the corresponding solutions are spectrally stable. For the hyperbolic case we consider the system where the mass in the middle is negligible; in this case the Eulerian-relative equilibria are unstable.

1 Introduction

The curved n-body problem has its origin in the papers written, independently, by Bolyai and Lovachevski [1, 15], the first discoverers of non-euclidean geometries in the 1830s. Lovachevski studied a Kepler problem in a three-dimensional hyperbolic space defining a special potential that extended the gravitational force proposed by Newton. Inspired by that paper, Killing proposed the same problem on a sphere of dimension three, obtaining a generalization of Kepler's three laws [11]. Other great mathematicians who worked on these kinds of problems were Schering [20] in 1870 and Lipschitz [14] in 1873, who proposed different potentials in the problem. Schering revised the paper of Lovachevski and obtained an analytic expression given by the cotangent potential, the same that we have used in this paper.

In 2005, Cariñena, Rañada, and Santander [13], working in the framework of differential geometry and taking as a reference the cotangent potential defined on S^2 and H^2, studied in a unified way the two body problem defined on spaces of nonzero constant curvature. The generalization for the n-body problem in spaces of constant curvature was obtained by F. Diacu, E. Pérez-Chavela, and M. Santoprete [7, 8]. In those papers the authors used the cotangent potential and the Euler-Lagrange equations

Received by the editors August 24, 2016; revised January 18, 2017.
Published electronically May 25, 2017.
The first author (EPC) has been partially supported by the Asociación Mexicana de Cultura A.C. The second one (JMSC) was partially supported by Conacyt México. This paper is part of Juan Manuel Sánchez Cerritos Ph.D. thesis in the Program: Doctorado en Ciencias (Matemáticas), at UAM-Iztapalapa, México.
AMS subject classification: **70F07**, 70G60.
Keywords: curved space, relative equilibrium, spectral stability.

with the corresponding constraints that maintain the particles on $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ (the model that we use for a surface of positive constant Gaussian curvature) embedded in $\mathbb{R}^3$, or H^2 seen as a sphere (with the Lorentzian metric) of imaginary radius -1 embedded in $\mathbb{R}^{2,1}$. We identify this with the upper part of the hyperboloid of two sheets $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = -1\}$ corresponding to the Weierstrass model of hyperbolic geometry, the model that we are using throughout this paper. In [4] you can find a nice description of the history of this fascinating problem.

Unlike the Newtonian case, the two body problem in spaces of constant curvature is non-integrable, as showed by Schepetilov in 1996 [19]. For $n \geq 2$, many researchers have worked in one of the simplest non trivial orbits that exist, the so-called relative equilibria, where the mutual distances among the masses remain constant for all time. For the case $n = 3$, Diacu, Pérez-Chavela, and Santoprete, [7], Diacu and Pérez-Chavela, [6], as well as R. Martínez and C. Simó, [12] got interesting results on Eulerian and Lagrangian solutions. Lagrangian orbits are given by equal masses forming an equilateral triangle. Martinez and Simó wrote one of the few papers in which the stability of Lagrangian periodic orbits has been studied on curved spaces.

To start our study we consider the motion of three point-particles moving on S^2 or H^2. Let q_i, $i = 1, 2, 3$, be the position of the i-th particle. The force function that extends from the Newtonian case to S^2 or H^2 is

$$U(q) = \sum_{i=1}^{3} \sum_{j=1, j\neq i}^{3} \frac{m_i m_j (\sigma)^{1/2} q_i * q_j}{\sqrt{\sigma - \sigma(q_i * q_j)^2}}$$

(see [3]), where $(*)$ denotes the usual scalar product in the case of positive curvature (that is, when we consider S^2), or the Lorentz product, denoted elsewhere by $\odot$, for negative curvature (that is, when we work on H^2); in the same way the symbol σ stands for $\sigma = 1$ if we consider S^2 or $\sigma = -1$ for H^2.

Using a variational method we can write the equations of motion as follows

$$\ddot{q}_i = \sum_{j\neq i}^{3} \frac{m_j [q_j - \sigma(q_i * q_j) q_i]}{[\sigma - \sigma(q_i * q_j)^2]^{3/2}} - \sigma(\dot{q}_i * \dot{q}_i) q_i, \quad i = 1, \ldots, n, \tag{1.1}$$

where $(\dot{\ })$ represents differentiation with respect to time t. The curved problem has energy and angular momentum as first integrals; however, the linear momentum is no longer a constant of motion in contrast with the Euclidean case [7]. In this paper we will study relative equilibria that are particular solutions of (1.1); the formal definition is the following.

Definition 1.1 A *relative equilibrium* is a solution of the curved n-body problem in which the mutual distances among the particles remain constant for all time $t \in \mathbb{R}$.

It is well known that for the Newtonian three body problem, any three arbitrary masses located at the vertices of an equilateral triangle generate a relative equilibrium called a *Lagrangian relative equilibrium* . In the curved three body problem with curvature $k > 0$ this kind of relative equilibria only exist if the three masses are equal; the stability of this kind of equilateral relative equilibria has been widely studied by

R. Martínez and C. Simó [12]. Another type of relative equilibria are the so-called *Eulerian-type relative equilibrium*, that is, relative equilibria in which the particles are, at each time t, on the same geodesic of the corresponding surface. Diacu, Pérez-Chavela, and Santoprete give a deep discussion of whether or not three masses, where two are equal and located at the ends of the configuration, generate a relative equilibrium. Recently, S. Zhu extended the results about those solutions and showed that any three positive masses can generate Eulerian-relative equilibria [21].

The stability of periodic orbits is one of the central problems in classical celestial mechanics. A lot of effort has been made to calculate and estimate limits of stability; see, for example, [2,16]. In the classical case the stability is fundamental to astrophysical processes, which is why we have focused on these kinds of problems. It is well known in the literature of dynamical systems that, in general, the analysis of the stability of equilibrium points of Hamiltonian systems is a really difficult task, in particular for relative equilibria seen as critical points for the system in rotating coordinates, usually we are only able to analyze their linear stability.

Definition 1.2 Consider the system $\dot{x} = f(x)$ and its linearization $\dot{x} = Ax$, where $A = Df(c)$ and c is an equilibrium solution. We say that the solution c is spectrally stable if the roots of the characteristic polynomial satisfy $\lambda^2 \leq 0$.

In general, when all masses in a relative equilibrium are on the same geodesic, we call it as an Eulerian relative equilibrium. In this paper we restrict our analysis to those Eulerian-relative equilibria in the three body problem where the two masses at the ends of the configuration are equal, the mass in the middle is arbitrary, and the geodesic distance between the central and the other masses is the same. Some authors call these kinds of relative equilibria *isosceles Eulerian-relative equilibria*; for brevity, we simply call them *Eulerian-relative equilibria*.

The main results of this paper correspond to the positive curvature case, where first, we give an open set of initial conditions on the parameters that assure spectral stability for the corresponding Eulerian-relative equilibria, showing that in this case the Lebesgue measure of this kind of periodic solution is positive. Then we go deeper in the analysis of the stability for all points in the parameter space, getting a total classification of the spectral stability for this kind of relative equilibrium. In other words, for all points in the parameter space, except for those that are on an algebraic curve, we can say if the corresponding relative equilibrium is spectrally stable or unstable. In the extended abstracts [17] we announced without proof a preliminary version of this result just for the case of three equal masses, the case for the two limit cases when the mass in the middle is negligible, and the case when the two equal masses at the ends are negligible. Since then, we have gone deeper in our research to get the material for this paper which is new and original.

For the case of negative curvature there are two kinds of Eulerian-relative equilibria, the so-called elliptic and hyperbolic relative equilibria studied early in [7]. In the first case we show that all Eulerian-relative equilibria are unstable, except for the particular degenerate case when the masses at the ends are negligible, where the corresponding relative equilibria are spectrally stable. For the hyperbolic case, due the complexity of the equations, we only did the analysis when the mass in the middle is negligible; in this case, the corresponding relative equilibria are unstable. This is in agreement with the results in [9], where the authors prove analytically that all hyperbolic relative equilibria in the general curved two body problem are unstable. A preliminary version of these results were stated without proof in the extended abstracts [17].

The paper is organized as follows. In Section 2 after the introduction we study the positive curvature case where we state and prove our main results Theorems 2.5 and 2.8. In Section 3 we study the negative curvature case, where we present some results about stability for elliptic and hyperbolic Eulerian relative equilibria.

2 The Positive Curvature Case

In the previous section we defined the relative equilibria in terms of mutual distances, that is, in terms of the isometric transformations. For the surfaces with positive curvature we use as a model the unitary sphere S^2, where it is well known that the geodesics are great circles and all isometries are uniform rotations in $\mathbb{R}^3$. Therefore, in this case, we can characterize the relative equilibria on S^2 in terms of the coordinates. This fact will facilitate the computations throughout the paper.

Proposition 2.1 *A solution $q_i, i = 1,2,3$, of the equations of motion on S^2 is a relative equilibrium if and only if $q_i = (x_i, y_i, z_i)$, with $x_i = r_i\cos(\omega t+\alpha_i)$, $y_i = r_i\sin(\omega t+\alpha_i)$, and $z_i =$ constant, where ω, α_i and $r_i, i = 1,2,3$, are constants, with $0 \le r_i = (1-z_i^2)^{1/2} \le 1$.*

This result is achieved from the principal axis theorem for $SO(3)$. Since all isometries in $\mathbb{R}^3$ are uniform rotations, we get that the relative equilibria are invariant solutions of the equations of motion under the group $SO(3)$. Now, the principal axis theorem states that any $A \in SO(3)$ can be written, in some orthonormal basis, as a rotation about a fixed axis. In the above proposition, without loss of generality, we have fixed the z-axis to obtain the result. In [18], the authors give a different proof using the stereographic projection.

Since the values of z_i remain constant along the motion, setting $\dot{z}_i = 0, \ddot{z}_i = 0$, in agreement with [7], we obtain the values of the angular velocity ω that maintain the same distances among the particles. We consider three positive point masses m_1, m_2, m_3 on the same geodesic of S^2; without loss of generality we assume that m_3 is fixed at $(0,0,1)$ and $m_1 = m_2$ are at the opposite sides of a diameter of the circle of radius $r = \sqrt{1-z^2}$ for a fix $z \in (-1,0)\cup(0,1)$. In this case $x_1 = r\cos(\omega t+\alpha)$, $y_1 = r\sin(\omega t+\alpha)$, $x_2 = r\cos(\omega t+\alpha+\pi)$, $y_2 = r\sin(\omega t+\alpha+\pi)$, $x_3 = 0$, $y_3 = 0$, that is, $r_1 = r_2 = r$ and $r_3 = 0$ (see Fig. 1 with $m_1 = m_2 = m$ and $m_3 = M$).

In order to simplify the computations we introduce the following time transformation and position coordinates; these kinds of transformations appeared also in [5,12]:

$$t = r^{\frac{3}{2}}\tau, \quad x_i = rX_i, \quad y_i = rY_i, \quad Q_i = (X_i, Y_i). \tag{2.1}$$

With these changes, the angular velocity becomes

$$\Omega^2 = \frac{m_1 + 4sm_3(1-r^2)}{4(1-r^2)^{\frac{3}{2}}},$$

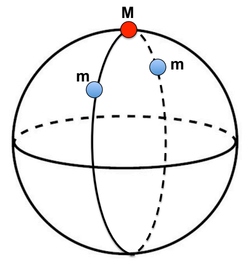


Figure 1. An Eulerian relative equilibrium on S^2.

where $s = 1$ if $z_1 = z_2 > 0$, or $s = -1$ if $z_1 = z_2 < 0$; Ω stands for $\frac{\omega t}{\tau}$ in the new coordinates.

Now we are able to express the system in a rotating frame, defining new variables (ξ_i, η_i), $i = 1,2,3$, as

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \mathcal{R}(\Omega\tau)\begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix},$$

where

$$\mathcal{R}(\Omega\tau) = \begin{pmatrix} \cos\Omega\tau & -\sin\Omega\tau \\ \sin\Omega\tau & \cos\Omega\tau \end{pmatrix}.$$

After tedious but straightforward computations by hand, it is possible to show that the new equations of motion are

$$\begin{aligned} \begin{pmatrix} \xi_i'' \\ \eta_i'' \end{pmatrix} &= 2\Omega\begin{pmatrix} \eta_i' \\ -\xi_i' \end{pmatrix} + \Omega^2\begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} - r^2 h_i \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} \\ &+ \sum_{j=1, j\neq i}^{3} m_j \big[\xi_i^2 + \eta_i^2 + \xi_j^2 + \eta_j^2 - s_{i,j} 2(\xi_i\xi_j + \eta_i\eta_j)T_{i,j} \\ &- r^2((\xi_i\xi_j + \eta_i\eta_j)^2 + (\xi_i^2 + \eta_i^2)(\xi_j^2 + \eta_j^2))\big]^{-3/2} \\ &\cdot \left[\begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix} - \big(r^2(\xi_i\xi_j + \eta_i\eta_j) + s_{i,j}T_{i,j}\big)\begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}\right], \end{aligned} \tag{2.2}$$

where

$$h_i = \Omega^2(\xi_i^2 + \eta_i^2) + 2\Omega(\xi_i\eta_i' - \eta_i\xi_i') + ((\xi_i'^2) + (\eta_i'^2)) + \frac{r^2}{1 - r^2(\xi_i^2 + \eta_i^2)}(\xi_i\xi_i' + \eta_i\eta_i')^2,$$

$$T_{i,j} = \sqrt{(1 - r^2(\xi_i^2 + \eta_i^2))(1 - r^2(\xi_j^2 + \eta_j^2))}, \qquad s_{i,j} = \operatorname{sign}(z_i z_j),$$

for $i = 1,2,3$. The symbol $(')$ stands for the derivative with respect τ.

The following lemma is helpful in our analysis; you can find a proof of it in [10].

https://doi.org/10.4153/CJM-2017-002-7 Published online by Cambridge University Press

Lemma 2.2 *Let* $p(x) = \frac{b_0}{b_3} + \frac{b_1}{b_3}x + \frac{b_2}{b_3}x^2 + x^3$, *and let* ρ_1, ρ_2, *and* ρ_3 *be its roots. The discriminant of the cubic polynomial* p *is given by*

$$d = \left(\frac{3\frac{b_1}{b_3} - (\frac{b_2}{b_3})^2}{9}\right)^3 + \left(\frac{9\frac{b_2}{b_3}\frac{b_1}{b_3} - 27\frac{b_0}{b_3} - 2(\frac{b_2}{b_3})^3}{54}\right)^2.$$

Suppose $d \leq 0$ *and* $b_0 \neq 0$. *Then* $\rho_1, \rho_2, \rho_3 < 0$ *if and only if* $\frac{b_0}{b_3}, \frac{b_1}{b_3}, \frac{b_2}{b_3} > 0$.

To analyze our problem let us normalize the total mass; *i.e.,* we define $m_T := m_1 + m_2 + m_3 = 1$. We consider $m_1 = m_2 = m$, and by the normalization of the total mass we have $m_3 = 1 - 2m$. The stability will depend on the masses and on the positions. As the factor r appears in even powers we denote $R := r^2$.

We define the parameter space $\mathcal{A}$ where we will be working by

$$\mathcal{A} = \left\{ (m, R) \in \left[0, \frac{1}{2}\right] \times (0,1) \right\}.$$

Notice that $R = 0$ and $R = 1$ correspond to a triple collision and to an antipodal singularity of the equations of motion, respectively.

Now we write the linearization of (2.2) for arbitrary masses $m_1 = m_2 = m$ where these particles are at height $z_1 = z_2 = z \neq 0$, and $m_3 = 1 - 2m$ is fixed at the point $(0,0,1)$. We can easily verify that, for any $z \neq 0$, the following values represent a fixed point in the rotating frame

$$\begin{array}{llll} \xi_1 = 1, & \eta_1 = 0, \xi_2 = -1, & \eta_2 = 0, \xi_3 = 0, & \eta_3 = 0, \\ \xi_1' = 0, & \eta_1', = 0, \xi_2' = 0, & \eta_2' = 0, \xi_3' = 0, & \eta_3' = 0. \end{array}$$

Let f be the vector field associated with system (2.2), seen as a first order system of differential equations at the fixed point. The linear part at the fixed point is given by

$$Df = \begin{pmatrix} 0 & I \\ A & \Omega B \end{pmatrix}, \tag{2.3}$$

where I is the identity matrix 6×6, and A and B are given by

$$B = \operatorname{diag}\left\{ \begin{pmatrix} 0 & 2-2R \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2-2R \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \right\},$$

$$A = \begin{pmatrix} a & 0 & b & 0 & c & 0 \\ 0 & d & 0 & e & 0 & f \\ b & 0 & a & 0 & c & 0 \\ 0 & e & 0 & d & 0 & f \\ -2m & 0 & -2m & 0 & g & 0 \\ 0 & m & 0 & m & 0 & h \end{pmatrix},$$

https://doi.org/10.4153/CJM-2017-002-7 Published online by Cambridge University Press

where a, b, c, d, e, f, g, and h are expressed as follows:

$$a = \Omega^2 - 3R\Omega^2 + s\frac{m + (8m-4)(R-2)}{4\sqrt{1-R}},$$

$$b = -\frac{1}{4}\frac{m(1-2R)}{(1-R)^{3/2}}, \qquad c = -2(1-2m)(1-R),$$

$$d = \frac{1}{8}\frac{m(2R-1)}{(1-R)^{3/2}} + \Omega^2 - R\Omega^2 + s(2m-1)\sqrt{1-R},$$

$$e = \frac{1}{8}\frac{m}{(1-R)^{3/2}}, \qquad f = 1-2m,$$

$$g = \Omega^2 + 4sm\sqrt{1-R}, \qquad h = \Omega^2 - 2sm\sqrt{1-R}.$$

Here $s = 1$ if $z > 0$, or $s = -1$ if $z < 0$.

The eigenvalues of Df are the zeroes of the polynomial $\det(Df - \lambda I) = 0$. Let us introduce μ such that $\lambda = \Omega\mu$. With this new variable the eigenvalue condition can be written as

$$\begin{pmatrix} 0 & I \\ A & \Omega B \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \Omega\mu\begin{pmatrix} u \\ v \end{pmatrix}.$$

If we denote the characteristic polynomial as $p(\mu)$, the eigenvalue condition must satisfy

$$p(\mu) = \det(A + \Omega^2\mu B - \Omega^2\mu^2 I) = 0. \tag{2.4}$$

2.1 The Three Equal Masses Case

In order to state and prove the main results of this paper, we start with the following result for the particular case where the three masses are equal. It will be very useful for the proof of the main theorem.

Lemma 2.3 *Consider Eulerian-relative equilibria of three masses moving on S^2, where m_1 and m_2 are at opposite ends of a diameter on the circle $z =$ constant $\neq 0$, and m_3 is fixed at $(0,0,1)$. If $m_1 = m_2 = m_3 = 1/3$, then the generated orbits are stable if $z \in (Z_1, Z_2)$, and unstable if $z \in (-\frac{1}{2}, 0) \cup (0, Z_1) \cup (Z_2, 1)$, where*

$$Z_1 = \frac{1}{2}\sqrt[4]{2},$$

$$Z_2 = \frac{1}{6}\frac{\sqrt{3}\sqrt{(27 + 3\sqrt{3}\sqrt{26})^{2/3} + 3}}{\sqrt[6]{27 + 3\sqrt{3}\sqrt{26}}}.$$

Eulerian-relative equilibria do not exist if $z \in (-1, -\frac{1}{2})$. At the points $z = -\frac{1}{2}$, $z = Z_1$, $z = Z_2$, it is not possible to conclude stability or instability. Notice that $z = 0$ corresponds to an antipodal singularity (see [7] for more details).

Proof We divide the proof into two cases, when m_1 and m_2 are in the northern hemisphere, and when they are in the southern hemisphere; we introduce the notation $\nu = \mu^2$.

https://doi.org/10.4153/CJM-2017-002-7 Published online by Cambridge University Press

When m_1 and m_2 are in the northern hemisphere, *i.e.*, when

$$z_1 = z_2 = z = (1-R)^{1/2} > 0,$$

the polynomial characteristic (2.4) in the variable v, after multiplying by

$$-(4096)(729)(1-R)^9/(4R-5)^3,$$

turns out to be

$$p(v) = v(v+1)Q_1(v)Q_2(v), \tag{2.5}$$

where

$$\begin{aligned} Q_1(v) &= 2R + 4vR - 5v - 5, \\ Q_2(v) &= (16R^2 - 40R + 25)v^3 + (32R^3 - 64R^2 + 18R + 15)v^2 \\ &\quad + (-256R^4 + 1040R^3 - 1528R^2 + 956R - 213)v \\ &\quad + 256R^5 - 1280R^4 + 2512R^3 - 2408R^2 + 1122R - 203. \end{aligned}$$

The factor $v = \mu^2$ in equation (2.5) comes out from the first integral associated with the energy. We recall from Definition 1.2 that a necessary and sufficient condition for spectral stability is that the eigenvalues of Df be either zero or purely imaginary, and hence, the solutions for v in equation (2.5) must be real and non-positive.

The explicit solution of $Q_1(v)$ is $v = \frac{5-2R}{4R-5} < 0$. Now, we analyze the factor $Q_2(v)$, which can be written as

$$Q_2(v) = g_1(R)v^3 + g_2(R)v^2 + g_3(R)v + g_4(R).$$

It is easy to check that $g_1(R) > 0$ for $R \in (0,1)$. We can also see that $g_2(R) > 0$ for $R \in (0,1)$, since both $g_2(0)$ and $g_2(1)$ are positive and the polynomial has a unique critical point at $R = \frac{2}{3} - \frac{1}{12}\sqrt{37} < 1$, which is a maximum on $(0,1)$.

For our purpose it is enough to verify that g_3 is positive in a certain interval. In this case it corresponds to the interval $(0.6, 0.8)$. This function has four real roots; one of them is at some $R < 0.6$, and the other three, which are different, are at some $R > 0.8$. Furthermore, in the interval $(0.6, 0.8)$ the function g_3 is positive. In order to verify these affirmations it is enough to see the following: $g_3(0) < 0$, $g_3(0.6) > 0$, $g_3(0.8) > 0$, $g_3(1) < 0$, $g_3(1.3) > 0$, $g_3(2) < 0$.

The function $g_4(R)$ has roots

$$\begin{aligned} R_1 &= -\frac{1}{12}(27 + 3\sqrt{78})^{\frac{1}{3}} - \frac{1}{4(27 + 3\sqrt{78})^{\frac{1}{3}}} + 1 \approx 0.61, \\ R_2 &= 1 - \frac{\sqrt{2}}{4} \approx 0.64, \qquad R_3 = 1 + \frac{\sqrt{2}}{4}, \\ R_{4,5} &= \frac{1}{24}(27 + 3\sqrt{78})^{\frac{1}{3}} + \frac{1}{8(27 + 3\sqrt{78})^{\frac{1}{3}}} + 1 \\ &\qquad \pm i\frac{\sqrt{3}}{4}\Big(-\frac{1}{6}(27 + 3\sqrt{78})^{\frac{1}{3}} + \frac{1}{2(27 + 3\sqrt{78})^{\frac{1}{3}}}\Big). \end{aligned}$$

We have that $g_4(R) > 0$ for $R \in (R_1, R_2)$. Here we can see that at $R = R_1$ and $R = R_2$, the polynomial $Q_2(v)$ has the value $v = 0$ as a root. At those points it is not possible to conclude the stability or instability of the orbits.

We have already verified that in the interval $(0,1) \setminus (R_1, R_2)$ the functions g_1 and g_2 are positive and g_4 is negative. With this we conclude that Q_2 has just one change of sign in its coefficients and therefore a positive root in this interval (Descartes' rule of signs). In this interval we have unstable orbits.

In the interval (R_1, R_2) the functions g_1, g_2, g_3, and g_4 are positive; this implies that (using again Descartes' rule of signs) in this interval $Q_2(v)$ has at least one negative root (because Q_2 is a polynomial of degree three). We are interested in showing that the other two are also negative.

According to Lemma 2.2, we need to see that the corresponding discriminant is non-positive.

The discriminant is (multiplied by 81 to avoid denominators)

$$\begin{aligned} d = &-12288R^{10} + 244736R^9 - 1632064R^8 + 5690848R^7 - 12117940R^6 \\ &+ 16855740R^5 - 15700889R^4 + 9737954R^3 - 3864429R^2 + 888300R - 89964. \end{aligned}$$

Its roots are (approximately)

$$\begin{aligned} r_1 &= 0.673119028735962, & r_2 &= 1.22978762865043, \\ r_3 &= 1.37154085389725, & r_4 &= 11.0351769580992, \\ r_{5,6} &= .6056086910 \pm i0.4335218317, & r_{7,8} &= 0.9505857618 \pm i0.3371311689, \\ r_{9,10} &= 1.247326646 \pm i0.4925143834. \end{aligned}$$

The polynomial d is negative for $R \in (-\infty, r_1)$, and we have $(R_1, R_2) \subset (-\infty, r_1)$. So we can conclude that $d < 0$ for $R \in (R_1, R_2)$, and, by Lemma 2.2, this implies that the three roots are real and negative.

Thus, for $z > 0$ we have stability if

$$\begin{aligned} z \in (Z_1, Z_2) := \left(\sqrt{1-R_2}, \sqrt{1-R_1}\right) &= \left(\frac{1}{2}\sqrt[4]{2}, \frac{1}{6}\frac{\sqrt{3}\sqrt{(27+3\sqrt{3}\sqrt{26})^{2/3}+3}}{\sqrt[6]{27+3\sqrt{3}\sqrt{26}}}\right) \\ &\approx (0.59, 0.61). \end{aligned}$$

Now, we analyze the case where m_1 and m_2 are in the southern hemisphere of S^2, $z < 0$. In this case, in order to have Eulerian-relative equilibria, the angular velocity with equal masses must satisfy

$$\Omega^2 = \frac{1 - 4(1-R)}{12(1-R)^{3/2}}.$$

This equation makes sense if $1 > R \geq \frac{3}{4}$. Therefore there are no Eulerian-relative equilibria if $-\frac{1}{2} \leq z < 0$ and the masses are equal (remember that $r^2 = R$ and $z < 0$).

For the corresponding values of $z_1, z_2 = z < 0$, and $z_3 = 1$, where Eulerian-relative equilibria exist, the factors $Q_1(v)$ and $Q_2(v)$ of the characteristic polynomial (2.5) take the form

$$
\begin{aligned}
Q_1(v) &= 6R + 4vR - 3v - 3,\\
Q_2(v) &= (16R^2 - 24R + 9)v^3 + (32R^3 - 64R^2 + 42R - 9)v^2\\
&\quad + (-256R^4 + 1008R^3 - 1512R^2 + 1020R - 261)v\\
&\quad + 256R^5 - 1280R^4 + 2544R^3 - 2520R^2 + 1242R - 243.
\end{aligned}
$$

The polynomial $Q_1(v)$ has root $v = \frac{3-6R}{4R-3} < 0$ for $R \in (\frac{3}{4}, 1)$. Let us analyze $Q_2(v)$; we can write it as

$$Q_2(v) = g_1(R)v^3 + g_2(R)v^2 + g_3(R)v + g_4(R).$$

The factor g_4 has roots $R_{1,2} = 1 \pm i\frac{\sqrt{2}}{4}$, $R_{3,4} = \frac{3}{4}$, $R_5 = \frac{3}{2}$, and we have $g_4(R) < 0$ for $R \in (\frac{3}{4}, 1)$. Here we can see that at $R = \frac{3}{4}$ $(z = -\frac{1}{2})$, Q_2 has the value $v = 0$ as a root, and at this point it is not possible to conclude stability or instability. The function g_3 has a global maximum at

$$R^* = -\frac{(417 + 96\sqrt{46})^{2/3} - 63\sqrt[3]{417 + 96\sqrt{46}} - 63}{64\sqrt[3]{417 + 96\sqrt{46}}} \approx 0.92,$$

and we have $g_3(R^*) < 0$. We have $g_1(R) > 0$ for $R \in (\frac{3}{4}, 1)$. Hence, independently of g_2 we can see that Q_2 has a change of sign for $R \in (\frac{3}{4}, 1)$, showing that Q_2 has a positive real root. This implies that, for $z_1, z_2 < 0$ in the corresponding interval, the orbits are unstable. With all above, Lemma 2.3 has been proved. ∎

2.2 The Limit Cases

In this section we are interested in showing the following result related to special values of the masses.

Proposition 2.4 *Consider Eulerian-relative equilibria of three masses moving on S^2, where m_3 is fixed at $(0, 0, 1)$, m_1 and m_2 are at opposite ends of a diameter on the circle $z =$ constant.*

(i) *If m_3 is negligible, and $m_1 = m_2$, the generated orbits are unstable for every $z \in (-1, 0) \cup (0, 1)$.*

(ii) *If m_1 and m_2 are negligible, then the generated orbits are stable if $z \in (0, 1)$, and Eulerian-relative equilibria do not exist if $z \in (-1, 0)$.*

Proof To show (i) we follow the same idea as in Lemma 2.3. Consider the equations (2.2) and its linearization (2.3) with $m_1 = m_2$ in the northern hemisphere and m_3 at the north pole. Taking into consideration its characteristic polynomial (2.4) written in the variable $v = \mu^2$, if we take $m \to \frac{1}{2}$ after multiply by $131072(1-R)^9$, this expression can be seen as

$$p(v) = v(v+1)(v + 2R + 1)Q_1(v)Q_2(v),$$

https://doi.org/10.4153/CJM-2017-002-7 Published online by Cambridge University Press

where

$$\begin{aligned} Q_1(\nu) &= \nu - 2R + 1, \\ Q_2(\nu) &= \nu^2 + \nu(-8R^2 + 16R - 6) - 128R^4 + 512R^3 - 760R^2 + 496R - 119 \\ &= \nu^2 + \nu h_1(R) + h_2(R). \end{aligned}$$

The factor Q_1 has root $\nu = -1 + 2R$, which is negative for $R \in (0, \frac{1}{2})$, so let us focus on roots of Q_2 in this interval. The factor $h_2(R)$ has roots $R_1 = 1 - \sqrt{2}/4 > 1/2$, $R_2 = 1 + \sqrt{2}/4$, $R_{3,4} = 1 \pm i/4$, and we have $h_2(R) < 0$ for $R \in (0, 1 - \sqrt{2}/4)$. Independently of the sign of h_1 we have a change of sign in the terms of Q_2. By Descartes' rule of signs, there is a positive root of $p(\nu)$. For $R \in (1 - \sqrt{2}/4, 1)$ the root of Q_1 is positive

If we consider the particles m_1 and m_2 in the southern hemisphere, then the corresponding characteristic polynomial takes the form (after multiplying by a factor that does not depend on ν)

$$p(\nu) = \nu(\nu + 1)(\nu + 2R + 1)Q_1(\nu)Q_2(\nu),$$

where

$$\begin{aligned} Q_1(\nu) &= \nu - 2R + 1, \\ Q_2(\nu) &= \nu^2 + (8R^2 - 16R + 10)\nu - 128R^4 + 512R^3 - 776R^2 + 528R - 135 \\ &= \nu^2 + \nu h_1(R) + h_2(R), \end{aligned}$$

As we did above, we analyze the roots of $Q_2(\nu)$ in the interval $R \in (0, 1/2)$. The factor $h_2(R)$ has roots $R_1 = 3/4$, $R_2 = 5/4$, $R_{3,4} = 1 \pm i\sqrt{2}/4$, and this function is negative in the interval $(-\infty, 3/4)$. Hence, again, by Descartes' rule there is a positive root of Q_2, and consequently a positive root of $p(\nu)$ as well. We are done with part (i) of the theorem.

To prove (ii) notice that if m_1 and m_2 are negligible, then there are no attraction forces between them, and each will move by the attraction generated by m_3. We have two decoupled systems that are symmetric. We will analyze the system consisting of particles m_1 and m_3.

If m_1 and m_2 are in the northern hemisphere, the angular velocity Ω satisfies (after considering $m \to 0$)

$$\Omega^2 = \frac{4(1-R)}{4(1-R)^{\frac{3}{2}}}.$$

The Jacobian matrix for this case is the 8×8 matrix given by

$$Df = \begin{pmatrix} 0 & I \\ A & \Omega B \end{pmatrix}$$

with

$$A = \begin{pmatrix} a & 0 & b & 0 \\ 0 & c & 0 & d \\ -2m & 0 & 2m(1-R)^{1/2} + \Omega^2 & 0 \\ 0 & m & 0 & \Omega^2 - m(1-R)^{1/2} \end{pmatrix},$$

https://doi.org/10.4153/CJM-2017-002-7 Published online by Cambridge University Press

where

$$a = \frac{R}{\sqrt{1-R}} + \Omega^2 - 3\Omega^2 R + 2\sqrt{1-R}, \quad b = 2R - 2,$$
$$c = \Omega^2 - \Omega^2 R - \sqrt{1-R}, \qquad d = 1.$$

The matrix B is given by

$$B = \operatorname{diag}\left\{ \begin{pmatrix} 0 & 2-2R \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \right\}.$$

The corresponding characteristic polynomial when we take $m \to 0$ is given by

$$p(\nu) = \frac{\nu(\nu+1)^3}{(R-1)^2}.$$

If we consider the particles $m_1 = m_2$ in the southern hemisphere and m_3 at $(0,0,1)$ then the expression of the angular velocity, in order to have Eulerian-relative equilibria, becomes (after taking $m \to 0$)

$$\Omega^2 = \frac{-4(1-R)}{4(1-R)^{\frac{3}{2}}},$$

which makes no sense, hence there are no Eulerian-relative equilibria when m_1 and m_2 are below the equator and negligible. This is enough to show the result (ii). ∎

2.3 The General Case

After the analysis for equal masses and the limit cases, now we are in a position to state the main results of this paper; the first one is the following theorem.

Theorem 2.5 *There exists an open subset $\mathcal{B}$ in the parameter space $\mathcal{A}$ with positive measure where the Eulerian-relative equilibria with parameters $(m, R) \in \mathcal{B}$ are spectrally stable.*

Proof Lemma 2.3 shows values of the masses and positions, $m_1 = m_2 = m_3 =: m^*$ (in the parameter space we have $m^* = 1/3$), $R \in (R_1, R_2) \subset (0,1)$, where the Euleria-relative equilibria are stable. In other words, values of m and R in the parameter space where the factor $Q_2(\nu)$ of the characteristic polynomial of Df,

$$Q_2 = (b_3\nu^3 + b_2\nu^2 + b_1\nu + b_0),$$

satisfies $b_0, b_1, b_2, b_3 > 0$, and its discriminant $d < 0$. Hence, by continuity of parameters, given any $R_0 \in (R_1, R_2)$ there exists an open set $\mathcal{B}$ of $(m^*, R_0) \in \mathcal{A}$ such that Q_2 also satisfies $b_0, b_1, b_2, b_3 > 0$, and $d < 0$ (Q_2 has three negative roots). The existence of this open set $\mathcal{B}$ is enough to conclude the proof of Theorem 2.5. ∎

Now we are going to analyze the stability in the complete parameter space $\mathcal{A}$.

We consider the equations of motion in the rotating frame (2.2), and its linearization, given by (2.3).

Consider first the case where the particles m_1 and m_2 are in the northern hemisphere. After multiplying by $-4096(1-R)^9/(8Rm-4R-7m+4)^3$, the characteristic polynomial takes the form

$$p(v) = v(v+1)Q_1(v)Q_2(v),$$

where

$$\begin{aligned}
Q_1 &= 8vRm - 4vR - 7vm + 4v + 10Rm - 4R - 7m + 4,\\
Q_2 &= (64R^2m^2 - 64R^2m - 112Rm^2 + 16R^2 + 120Rm + 49m^2 - 32R - 56m + 16)v^3\\
&\quad + \big(-64R^3m^2 + 32R^3m + 296R^2m^2 - 216R^2m - 378Rm^2 + 32R^2 + 324Rm\\
&\qquad + 147m^2 - 64R - 140m + 32\big)v^2\\
&\quad + (128R^4m^2 - 128R^4m - 496R^3m^2 + 512R^3m + 752R^2m^2 - 808R^2m\\
&\quad - 532Rm^2 + 16R^2 + 592Rm + 147m^2 - 32R - 168m + 16)v\\
&\quad + 256R^5m^2 - 896R^4m^2 - 128R^4m + 1072R^3m^2 + 480R^3m - 440R^2m^2\\
&\quad - 656R^2m - 42Rm^2 + 388Rm + 49m^2 - 84m.
\end{aligned}$$

In order to analyze the roots of the characteristic polynomial, $p(M)$, we will do this for Q_1 and Q_2.

Lemma 2.6 *If $Q_1(v_0) = 0$, then $v_0 < 0$.*

Proof We can see that $Q_1(v_0) = 0$ if and only if

$$v_0 = -\frac{10Rm - 4R - 7m + 4}{8Rm - 4R - 7m + 4} =: f(m,R).$$

Then

$$\frac{\partial f}{\partial m} = 8\frac{R(R-1)}{(8Rm - 4R - 7m + 4)^2} < 0$$

for $(m,R) \in \mathcal{A}$. Hence f does not have a critical point. The maximum and minimum values should be on the border of the region. We can see that $f(m,0) = f(0,R) = -1$, $f(m,1) = -3$, and $f(\frac{1}{2},R) = -2R-1 < 0$. Hence, we conclude that $f(m,R) = v_0 < 0$. ∎

We now study the behavior of the roots of $Q_2(v)$; they will give us the stability region for the relative equilibria.

Now let us focus on $Q_2(v)$, this expression is a cubic polynomial on v with coefficients depending on m and R, $Q_2(v) = b_0 + b_1v + b_2v^2 + b_3v^3$. The discriminant d of $Q_2(v)$ has the form

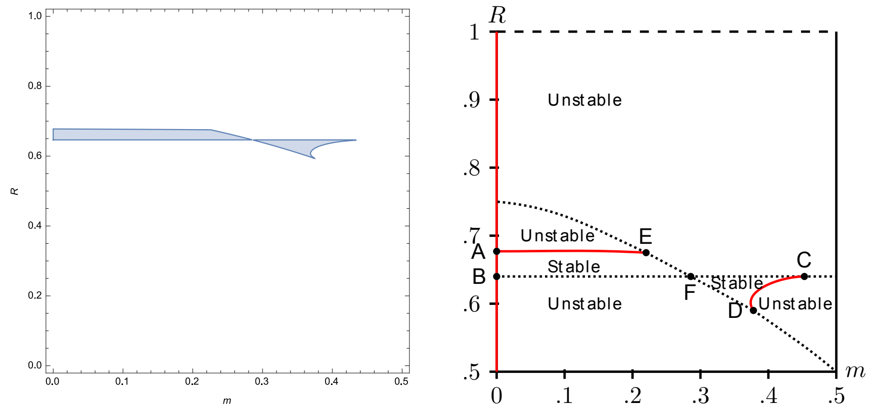
$$d = \frac{64}{27}\frac{m^2(R-1)^2D_1}{(8Rm - 4R - 7m + 4)^6}, \tag{2.6}$$

where D_1 can be seen in the appendix.

It is possible to see that b_3 is positive for $(m,R) \in \mathcal{A}$, so we can focus on the sign of b_0, b_1, and b_2 to analyze the stability (instead of $\frac{b_0}{b_3}$, $\frac{b_1}{b_3}$, and $\frac{b_2}{b_3}$).

Lemma 2.7 *For $(m,R) \in \mathcal{A}$, $b_3 > 0$.*

https://doi.org/10.4153/CJM-2017-002-7 Published online by Cambridge University Press



(a) Stability region $\mathcal{B}$ in the rectangle $(m, R) \in [0, 1/2] \times (0, 1)$.

(b) Border $\partial\mathcal{B}$, including curves where there are zero-eigenvalues ($b_0 = 0$, dashed curves).

Figure 2. Bifurcation diagram.

Proof Let us prove that b_3 is positive by checking that there are no minima nor maxima inside $\mathcal{A}$. Once we have done this we notice that on $\partial\mathcal{A}$, we have $b_3 > 0$. We compute $\partial b_3/\partial R = (128\,R - 112)m^2 + (-128\,R + 120)m + 32\,R - 32$. Hence, $\partial b_3/\partial m = 0$ if and only if $m = 4\,(R-1)/(8\,R-7) := f(R)$.

Now we have to see if there are values of $m \in [0, \frac{1}{2}]$ that satisfy $f(R) = m$ for $R \in (0,1)$. We have $f'(R) = 4/(8R-7)^2 > 0$. By noticing that $f(0) > \frac{1}{2}$ and $f(1) = 0$, we can conclude that there are no values of $m \in [0, \frac{1}{2}]$ that satisfies $f(R) = m$ for $R \in (0,1)$. Then there are no any critical points in $\mathcal{A}$. Furthermore, we can check that on $\partial\mathcal{A}$, $b_3 > 0$, since we have $b_3(0, R) = 16(R-1)^2$, $b_3(\frac{1}{2}, R) = \frac{1}{4}$, $b_3(m, 0) = (7m-4)^2$, and $b_3(m, 1) = m^2$. Hence, $b_3 > 0$. ■

The algorithm to analyze the stability is as follows: for points on $\mathcal{A}$ we compute the discriminant d, if it is positive, then the corresponding orbit is unstable, otherwise we compute b_0. If $b_0 = 0$, then we cannot conclude stability of the orbits with the corresponding parameters. If $b_0 < 0$, $b_1 < 0$, and $b_2 < 0$, then the orbits are stable, otherwise they are unstable. Figure 2 shows the parameter-region $\mathcal{B}$ where the orbits are stable.

At this point we have analyzed the case where the three particles are above the equator; now let us study the case where two of them are below this great circle. Hence, consider the masses m_1 and m_2 in the southern hemisphere.

Recall that the angular velocity in this case should satisfy

$$\Omega^2 = \frac{m - 4(1-2m)(1-R)}{4(1-R)^{\frac{3}{2}}}.$$

The above equation makes sense if the numerator is non-negative, which is analogous to the condition $\frac{4-9m}{4-8m} \le R$ (see Figure 3).

https://doi.org/10.4153/CJM-2017-002-7 Published online by Cambridge University Press

The corresponding characteristic polynomial, after multiplying by

$$-4096(1-R)^9/(8\,Rm-4\,R-9\,m+4)^3,$$

has the form

$$p(\mu)=p(\nu)=\nu(\nu+1)Q_1(\nu)Q_2(\nu),$$

where

$$\begin{aligned}
Q_1(\nu) &= 8\nu Rm-4\nu R-9\nu m+4\nu+6\,Rm-4\,R-9\,m+4,\\
Q_2(\nu) &= \big(64\,R^2m^2-64\,R^2m-144\,Rm^2+16\,R^2+136\,Rm+81\,m^2\\
&\quad -32R-72\,m+16\big)\nu^3\\
&\quad +\big(-64\,R^3m^2+32\,R^3m+344\,R^2m^2-232\,R^2m-522\,Rm^2+32\,R^2\\
&\quad +380\,Rm+243\,m^2-64\,R-180\,m+32\big)\nu^2\\
&\quad +\big(128\,R^4m^2-128\,R^4m-528\,R^3m^2+512\,R^3m+912\,R^2m^2\\
&\quad -856\,R^2m-756\,Rm^2+16\,R^2+688\,Rm+243\,m^2-32\,R-216\,m+16\big)\nu\\
&\quad +256\,R^5m^2-896\,R^4m^2-128\,R^4m+1104\,R^3m^2+480\,R^3m-456\,R^2m^2\\
&\quad -688\,R^2m-90\,Rm^2+444\,Rm+81\,m^2-108\,m.
\end{aligned}$$

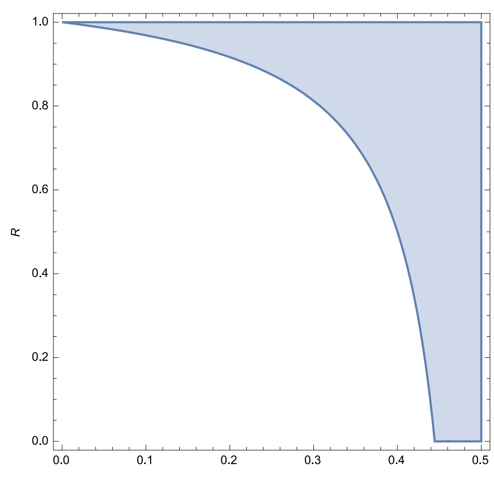


Figure 3. Region in the parameter space where Eulerian-relative equilibria exist if $z_1=z_2<0$.

The discriminant d of $Q_2(\nu)$ has the form

$$d=\frac{64}{27}\frac{m^2(R-1)^2D_2}{(8Rm-4R-9m+4)^6}, \tag{2.7}$$

where D_2 can be seen in the appendix.

https://doi.org/10.4153/CJM-2017-002-7 Published online by Cambridge University Press

We use the same algorithm to check the points in the parameter space to decide the stability or instability. After taking the algorithm we can see that for any points in the parameter space where Eulerian-relative equilibria exist, these solutions are unstable.

With all the above facts we have proved the second main result of this work, which we state as follows.

Theorem 2.8 *Consider masses $m_1 = m_2 =: m$ and $m_3 =: M$, where m_1 and m_2 are at opposite ends of a diameter on $z \neq 0$, and m_3 is fixed at $(0,0,1)$. Consider also, if it exists, the Eulerian-relative equilibrium generated by these masses and positions. For any parameters (m, M, z) it is possible to decide the stability or instability of the corresponding Eulerian-relative equilibrium, except for a zero-measure set.*

2.3.1 Bifurcations

The bifurcations take place in $\partial\mathcal{B}$ and on a not-connected set on the line $m = 0$. In this section we analyze the behavior of the polynomial $Q_2(v)$ with parameters in this set.

The set $\partial\mathcal{B}$ is composed of subsets in the parameter space where $d = 0$, $b_0 = 0$ and $b_1 = 0$ (see Figure 2b). We have $b_2 > 0$ on $\mathcal{B} \cup \partial\mathcal{B}$. This fact means that the polynomial does not have zero as a triple root for any pair (m, R).

There are six interesting points in $\partial\mathcal{B}$, which are points are the vertices of $\mathcal{B}$:

$$A = (0, 0.6777905), \qquad B = \left(0, 1 - \frac{1}{4}\sqrt{2}\right),$$
$$C = \left(0.45308704, 1 - \frac{1}{4}\sqrt{2}\right), \qquad D = (0.378794, 0.5906987),$$
$$E = (0.225915, 0.675279), \qquad F = \left(\frac{2}{7}, 1 - \frac{1}{4}\sqrt{2}\right),$$

(with error 10^{-10}), see figure 2b. At point A we have $d = 0$ and $b_0 = 0$; at point B, $b_0 = 0$; points C and D represent the parameters where $d = 0$, $b_0 = 0$, and $b_1 = 0$. At point E, we have $d = 0$ and $b_0 = 0$. At point F we have $b_0 = 0$. The coefficient $b_1 > 0$ on $\mathcal{B} \cup \partial\mathcal{B}$ except at points C and D; at those points $b_1(C), b_1(D) = 0$.

The behavior of the polynomial $Q_2(v)$ with parameters represented by the vertices is as follows (see Figure 4).

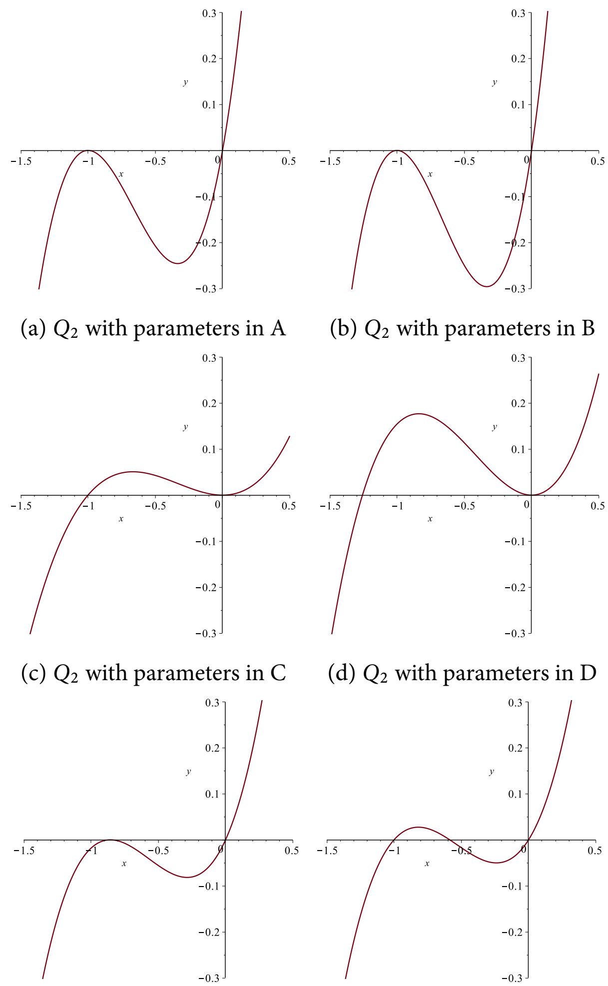
- Parameters on C and D: the polynomial has a negative root and zero is a double root.
- Parameters on E: the polynomial has zero as a simple root and a negative number as a double root.
- Parameters on F: the polynomial has two different negative roots (one of them being -1), and zero is also a root.
- Parameters along the line $m = 0$ (including points A and B), the polynomial has -1 as a double root, and zero is also a root.

We know the behavior of $Q_2(v)$ on the vertices of $\mathcal{B}$; then by continuity of the parameters we can explain the behavior of the polynomial along $\partial\mathcal{B}$.

(a) On the curve $A - E$ the polynomial has zero as a simple root and a negative number as a double root.

(b) On the curve $E - D$ the polynomial has zero as a simple root and a negative double root on E and as the parameters goes to the point D, Q_2 has zero and two different negative roots, until the parameters reach the point D, where Q_2 has zero as a double root and one negative simple root.

(c) On the curve $C - D$, the polynomial goes from parameters on the point C, where it has zero as a double root and a negative root, to the point D, where Q_2 has



(a) Q_2 with parameters in A (b) Q_2 with parameters in B

(c) Q_2 with parameters in C (d) Q_2 with parameters in D

(e) Q_2 with parameters in E (f) Q_2 with parameters in F

Figure 4. Polynomial $Q_2(v)$ with parameters on the vertices of the bifurcation diagram

https://doi.org/10.4153/CJM-2017-002-7 Published online by Cambridge University Press

also zero as a double root and a negative root, going through points on the curve where the polynomial has a negative double root and a different negative simple root.

(d) On the segment of line $B - C$ the polynomial has zero as a simple root, and it has two different negative roots on all the points along the segment, except on the starting and ending points.

Remark 2.9 The set where Q_2 has zero roots is a zero-measure set, $\mathcal{C}$, *i.e.*,

$$\begin{aligned}\mathcal{C} &:= \{(m,R) \in \mathcal{A} | b_0 = 0\}\\ &= \Big\{(m,R) \in \mathcal{A} | R = 1 - \frac{1}{4}\sqrt{2},\Big\} \cup \{(m,R) \in \mathcal{A} | m = 0\}\\ &\quad \cup \Big\{(m,R) \in \mathcal{A} | m = 4\,\frac{4R^2 - 7R + 3}{32R^3 - 48R^2 + 10R + 7}\Big\}.\end{aligned}$$

Notice that the line $m = 0$ corresponds to Proposition 2.4(ii), and when the parameters lies on that segment we have stability of the orbits. Then the case where it is not possible to say something about stability is in the set

$$\Big\{(m,R) \in \mathcal{A} \mid R = 1 - \frac{1}{4}\sqrt{2},\Big\} \cup \Big\{(m,R) \in \mathcal{A} | m = 4\,\frac{4R^2 - 7R + 3}{32R^3 - 48R^2 + 10R + 7}\Big\},$$

which is represented by the dashed lines on Figure 2b.

3 The Negative Curvature Case

In this section we prove some results about stability of Eulerian-relative equilibria in H^2. Using the principal axis theorem it is possible to show in explicit form the coordinates of relative equilibria solutions. We will see that we can define three kinds of relative equilibria: elliptic, hyperbolic, and parabolic, but Diacu, Pérez-Chavela, and Santoprete proved the non-existence of the parabolic-relative equilibria in [7].

Proposition 3.1 *A solution q_i, $i = 1,2,3$, of the equations of motion on H^2 is a relative equilibrium, if and only if $q_i = (x_i, y_i, x_i)$, where the coordinates are given in one of the following forms*

- $x_i = r_i\cos(\omega t + \alpha_i)$, $y_i = r_i\sin(\omega t + \alpha_i)$, $z_i = \text{constant}$, *where ω, α_i and $r_i = (z_i^2 - 1)^{1/2} \in (0,\infty)$, $i = 1,2,3$, are constants.*
- $x_i = \text{constant}$, $y_i = \rho_i\sinh(\omega t + \alpha_i)$, $z_i = \rho_i\cosh(\omega t + \alpha_i)$, *where ω, α_i and $\rho_i = (1 + x_i^2)^{1/2} > 1$, $i = 1,2,3$, are constants.*

To check this result we can see, as in S^2, that relative equilibria are solutions of (1.1) generated by orthogonal transformations of determinant ± 1 that leave H^2 invariant; this is a closed group called the *Lorentz group*, $\mathrm{Lor}(\mathbb{R}^{2,1}, \odot)$. First we define a Lorentzian trajectory about an axis as the 1-parameter subgroup of $\mathrm{Lor}(\mathbb{R}^{2,1}, \odot)$ leaving the axis point-wise fixed. The principal axis theorem in this case states that every

https://doi.org/10.4153/CJM-2017-002-7 Published online by Cambridge University Press

$G \in \mathrm{Lor}(\mathbb{R}^{2,1}, \odot)$ has one of the following canonical forms:

$$A = P\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1}, \qquad B = P\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh s & \sinh s \\ 0 & \sinh s & \cosh s \end{pmatrix} P^{-1},$$

$$C = P\begin{pmatrix} 1 & -t & t \\ t & 1-\frac{t^2}{2} & \frac{t^2}{2} \\ t & -\frac{t^2}{2} & 1+\frac{t^2}{2} \end{pmatrix} P^{-1},$$

where $\theta \in [0, 2\pi)$, $s, t \in \mathbb{R}$, and $P \in \mathrm{Lor}(\mathbb{R}^{2,1}, \odot)$.

The above transformations are called elliptic, hyperbolic, and parabolic, respectively. To match this standard terminology, from this point we call the trajectories generated by these transformations: elliptic-relative equilibria, hyperbolic-relative equilibria, and parabolic-relative equilibria, but as we have mentioned before, parabolic equilibria do not exist.

In H^2, as in S^2, it is possible to find values of angular velocity Ω in terms of the masses that produce elliptic and hyperbolic-relative equilibria [7]. Given these values of Ω we are able to study the stability of the generated orbits.

To achieve our goal we will follow the same idea as in S^2.

3.1 Elliptic Eulerian-Relative Equilibria

We consider three point particles with positive masses m_1, m_2, m_3 with $m_3 = M$ fixed at the point $(0,0,1)$ and $m_1 = m_2 = m$ are at the opposite sides of a diameter of the circle of radius $r = \sqrt{1-z^2}$ for a fix $z \in (1, \infty)$. As for the analysis of the relative equilibria on S^2 studied in Section 2, here $x_1 = r\cos(\omega t + \alpha)$, $y_1 = r\sin(\omega t + \alpha)$, $x_2 = r\cos(\omega t + \alpha + \pi)$, $y_2 = r\sin(\omega t + \alpha + \pi)$, $x_3 = 0$, $y_3 = 0$, that is, $r_1 = r_2 = r$ and $r_3 = 0$ (see Fig. 5).

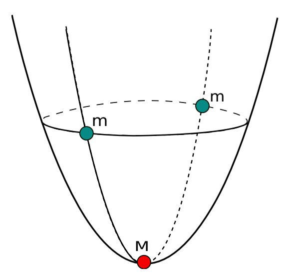


Figure 5. An elliptic Eulerian relative equilibrium on H^2.

Consider also the same transformations as in S^2, (2.1). Now we express the system in a rotating frame with variables (ξ_i, η_i), $i = 1, 2, 3$ as

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \mathcal{R}(\Omega\tau)\begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix},$$

https://doi.org/10.4153/CJM-2017-002-7 Published online by Cambridge University Press

where $R(\Omega\tau)$ is the matrix which represents the elliptic transformations

$$\mathcal{R}(\Omega\tau) = \begin{pmatrix} \cos\Omega\tau & -\sin\Omega\tau \\ \sin\Omega\tau & \cos\Omega\tau \end{pmatrix}.$$

After a straightforward computation the new equations of motion are expressed as

$$\begin{aligned}
\text{(3.1)} \qquad \begin{pmatrix} \xi_i'' \\ \eta_i'' \end{pmatrix} &= 2\Omega \begin{pmatrix} \eta_i' \\ -\xi_i' \end{pmatrix} + \Omega^2 \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} + r^2 h_i \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} \\
&+ \sum_{j=1, j\neq i}^{3} m_j \Big[\xi_i^2 + \eta_i^2 + \xi_j^2 + \eta_j^2 - 2(\xi_i\xi_j + \eta_i\eta_j) T_{i,j} \\
&\quad + r^2 ((\xi_i\xi_j + \eta_i\eta_j)^2 + (\xi_i^2 + \eta_i^2)(\xi_j^2 + \eta_j^2)) \Big]^{-3/2} \\
&\quad \cdot \left[\begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix} + (r^2(\xi_i\xi_j + \eta_i\eta_j) - T_{i,j}) \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} \right],
\end{aligned}$$

where

$$h_i = \Omega^2(\xi_i^2 + \eta_i^2) + 2\Omega(\xi_i\eta_i' - \eta_i\xi_i') + ((\xi_i'^2) + (\eta_i'^2)) - \frac{r^2}{1 + r^2(\xi_i^2 + \eta_i^2)}(\xi_i\xi_i' + \eta_i\eta_i')^2,$$

$$T_{i,j} = \sqrt{(1 + r^2(\xi_i^2 + \eta_i^2))(1 + r^2(\xi_j^2 + \eta_j^2))},$$

for $i = 1, 2, 3$.

In this section our goal is to show the following result.

Theorem 3.2 *Consider Eulerian-relative equilibria of three masses moving on H^2, where $m_1 = m_2$ are at opposite ends of a diameter on the circle $z =$ constant, and m_3 is fixed at $(0,0,1)$. These solutions are unstable for any $z > 1$ and any value of masses.*

As in S^2 let us normalize the total mass, $m_1 + m_2 + m_3 = 1$. Set $m_1 = m_2 = m$; then we have $m_3 = 1 - 2m$.

It is easy to check that

$$\begin{array}{llll} \xi_1 = 1, & \eta_1 = 0, \xi_2 = -1, & \eta_2 = 0, \xi_3 = 0, & \eta_3 = 0, \\ \xi_1' = 0, & \eta_1', = 0, \xi_2' = 0, & \eta_2' = 0, \xi_3' = 0, & \eta_3' = 0, \end{array}$$

is a fixed point in the rotating frame. Following the same idea as in S^2, the differential of the vector field f corresponding to (3.1), and setting $\nu = \mu^2$, it is possible to show that the characteristic polynomial, after multiplying by

$$\frac{4096}{(7m - 4R + 8mR - 4)^3(8\nu^2 Rm - 4\nu^2 R + 7\nu^2 m - 4\nu^2 + 10Rm - 4R + 7m - 4)},$$

is given by

$$p(\nu) = \nu(\nu + 1)Q_1(\nu)Q_2(\nu),$$

where, as in the analysis for the relative equilibria on S^2 studied in Section 2, we have done $R = r^2$ and

$$\begin{aligned}
Q_1(v) =& (8\,Rm - 4\,R + 7\,m - 4)v + 10\,Rm - 4\,R + 7\,m - 4,\\
Q_2(v) =& \big(64\,R^2m^2 - 64\,R^2m + 112\,Rm^2 + 16\,R^2 - 120\,Rm\\
&+ 49\,m^2 + 32\,R - 56\,m + 16\big)v^3\\
&+ \big(64\,R^3m^2 - 32\,R^3m + 296\,R^2m^2 - 216\,R^2m + 378\,Rm^2 + 32\,R^2\\
&- 324\,Rm + 147\,m^2 + 64\,R - 140\,m + 32\big)v^2\\
&+ \big(128\,R^4m^2 - 128\,R^4m + 496\,R^3m^2 - 512\,R^3m + 752\,R^2m^2\\
&- 808\,R^2m + 532\,Rm^2 + 16\,R^2 - 592\,Rm + 147\,m^2 + 32\,R - 168\,m + 16\big)v\\
&- 256\,R^5m^2 - 896\,R^4m^2 - 128\,R^4m - 1072\,R^3m^2 - 480\,R^3m\\
&- 440\,R^2m^2 - 656\,R^2m + 42\,Rm^2 - 388\,Rm + 49\,m^2 - 84\,m.
\end{aligned}$$

The polynomial $Q_2(v)$ can be seen as $Q_2(v) = b_3v^3 + b_2v^2 + b_1v + b_0$. The parameter space is $\mathcal{C} = \{(m, R) \in [0, \frac{1}{2}] \times (0, \infty)\}$.

In order to conclude the proof of the above theorem, we will use the signs of the coefficients b_3 and b_0, given in the following two lemmas.

Lemma 3.3 *For $(m, R) \in (0, \frac{1}{2}] \times (0, \infty)$, $b_3 > 0$.*

Proof In order to verify that $b_3 > 0$, we will see that there is no critical point inside the region $(0, \frac{1}{2}] \times (0, \infty)$. We compute

$$\frac{\partial b_3}{\partial R} = 8(-1 + 2\,m)(8\,Rm - 4\,R + 7\,m - 4).$$

We will now see that the factor $(8\,Rm - 4\,R + 7\,m - 4) := g(m, R)$ is negative. For this we compute $\partial g/\partial R = 8m - 4 < 0$ for $m \in (0, \frac{1}{2}]$, and we have $g(m, 0) = 7m - 4 < 0$. With these simple calculations we conclude that $g(m, R) < 0$ for $(m, R) \in (0, \frac{1}{2}] \times (0, \infty)$.

Hence $\partial b_3/\partial R > 0$. Knowing this and with the fact that $b_3(0, m) = (7m - 4)^2 > 0$, we conclude that $b_3 > 0$. For $m = \frac{1}{2}$, $b_3 = \frac{1}{4} > 0$. This completes the proof. ■

We now focus on b_0.

Lemma 3.4 *For $(m, R) \in (0, \frac{1}{2}] \times (0, \infty)$, $b_0 < 0$.*

Proof It is easy to see that we can write $42Rm^2 - 388Rm = 2Rm(21m - 194)$, and this term is negative for $m \in (0, 1/2]$. Substituting this in b_0 we can easily check that $b_0 < 0$. ■

With the above two lemmas we conclude the proof of Theorem 3.2.

If $m = 0$, then $b_0 = 0$, so with the above method it is not possible to conclude anything about stability, because a new zero eigenvalue appears. This problem can be seen as two symmetric decoupled problems. We show this case in the following proposition.

Proposition 3.5 *Consider Eulerian-relative equilibria of three masses moving on H^2, where m_1 and m_2 are at opposite ends of a diameter on the circle $z = \text{constant}$, and m_3 is fixed at $(0,0,1)$. If m_1 and m_2 are negligible, then the generated orbits are stable if $z \in (1,\infty)$.*

Proof In this case the particles m_1 and m_2 will be moving independently by the force generated by m_3. As we mentioned before, we have two symmetric decoupled problems. We will analyze the system consisting of particles m_1 and m_3. Consider masses $m_1 = m$ and $m_3 = 1$.

The corresponding matrix Df has characteristic polynomial

$$p(\mu) = -\frac{F}{256(R+1)^6}, \tag{3.2}$$

where F can be seen in the appendix. When m is negligible, taking $m \to 0$ we get

$$p(\mu) = \frac{\mu^2(\mu^2+1)^3}{(R+1)^2},$$

showing that there are only purely imaginary eigenvalues. ∎

3.2 Hyperbolic Relative Equilibria

In this section we consider three point particles m_1, m_2, and m_3 moving along geodesics. The particle m_3 moving on the geodesic with coordinate $x_3 = 0$, and $m_1 = m_2$ moving symetrically on hyperbolas satisfying $x_1 = -x_2 = x = \sqrt{\rho^2 - 1} = \text{constant} > 0$.

Consider the transformations (2.1), with ρ instead of r. We express the system in a rotating frame with variables (ξ_i, η_i), $i = 1,2,3$ as

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \mathcal{R}(\Omega\tau)\begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix},$$

where $R(\Omega\tau)$ is the matrix that represents the hyperbolic transformations

$$\mathcal{R}(\Omega\tau) = \begin{pmatrix} \cosh\Omega\tau & \sinh\Omega\tau \\ \sinh\Omega\tau & \cosh\Omega\tau \end{pmatrix}.$$

After a straightforward computation we obtain the equations of motion in the new rotating frame.

$$\begin{aligned} \begin{pmatrix} \xi_i'' \\ \eta_i'' \end{pmatrix} &= -2\Omega\begin{pmatrix} \eta_i' \\ \xi_i' \end{pmatrix} - \Omega^2\begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} + \rho^2 h_i \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix} \\ &\quad + \sum_{j=1, j\neq i}^{3} m_j \Big[\, \xi_i^2 - \eta_i^2 + \xi_j^2 - \eta_j^2 - 2(\xi_i\xi_j - \eta_i\eta_j)T_{i,j} \\ &\qquad + \rho^2\big((\xi_i\xi_j - \eta_i\eta_j)^2 + (\eta_i^2 - \xi_i^2)(\eta_j^2 - \xi_j^2)\big)\Big]^{-3/2} \\ &\qquad \cdot \left[\begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix} + \big(\rho^2(\xi_i\xi_j - \eta_i\eta_j) - T_{i,j}\big)\begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}\right], \end{aligned} \tag{3.3}$$

where

$$h_i = \Omega^2(\eta_i^2 - \xi_i^2) + 2\Omega(\eta_i \xi_i' - \xi_i \eta_i') + ((\xi_i'^2) - (\eta_i'^2)) + \frac{\rho^2}{\rho^2(\eta_i^2 - \xi_i^2) - 1}(\eta_i \eta_i' - \xi_i \xi_i')^2,$$

$$T_{i,j} = \sqrt{(\rho^2(\eta_i^2 - \xi_i^2) - 1)(\rho^2(\eta_j^2 - \xi_j^2) - 1)},$$

for $i = 1, 2, 3$.

It is possible to see that $\xi_1 = 0,\ \eta_1 = 1,\ \xi_2 = 0,\ \eta_2 = -1,\ \xi_3 = 0,\ \eta_3 = 0,\ \xi_1' = 0,\ \eta_1', = 0,\ \xi_2' = 0,\ \eta_2' = 0,\ \xi_3' = 0,\ \eta_3' = 0$ is a fixed point of the system.

We will prove the following result about hyperbolic Eulerian-relative equilibria.

Theorem 3.6 *Consider Eulerian-relative equilibria of three masses moving on H^2, where the body m_i has coordinates $q_i = (x_i, y_i, z_i)$. If m_3 is negligible and is on the geodesic contained in the yz-plane, and m_1 and m_2 are on hyperbolas of the form $x_1 =$ constant, $x_2 = -x_1$, $y_1 = y_2 = \rho \sinh(t)$, $z_1 = z_2 = \rho \cosh(t)$, $t \in \mathbb{R}$, then the generated orbits are unstable for any $\rho > 1$.*

Proof Consider masses $m_1 = m_2 = 1$ and $m_3 = m$. The corresponding matrix of the differential of the vector field related to the system (3.3) at the fixed point has characteristic polynomial $p(\mu)$ that depends on the variables μ, R and m (in a similar way to S^2, R stands for ρ^2). If we take $m \to 0$ in the characteristic polynomial, we have

$$\begin{aligned} p(\mu) = &\frac{1}{1024(R-1)^9}\mu^2 \cdot (\mu+1)\cdot(\mu-1) \\ &\cdot\Big[\Big(2\sqrt{2}+\frac{5}{2}\Big)\mu^2 - 2\sqrt{2} - R\,(4\sqrt{2}+5) - \frac{5}{2}\Big] \\ &\cdot\big[\mu^4 + (3\sqrt{2}-2)\,\mu^2 + (-4\sqrt{2}-5)\,R^2 + (8-3\sqrt{2})\,R + 3\sqrt{2}+1\big] \\ &\cdot\Big(\Big(\frac{2\sqrt{2}}{7} - \frac{5}{14}\Big)\mu^2 - \frac{2\sqrt{2}}{7} + R\Big(\frac{4\sqrt{2}}{7} - \frac{5}{7}\Big) + \frac{5}{14}\Big). \end{aligned}$$

The factor $(\mu-1)$ shows a real positive root of the characteristic polynomial; hence we conclude our result. ∎

4 Appendix

In this appendix we give some expressions used throughout the work. All computations have been done using the algebraic manipulators of Matlab and checked again using Maple.

The following expression is for the factors of the discriminant of equation (2.6):

$D_1 =$

$602112\,R^{10}m^4 - 745472\,R^{10}m^3 - 4498432\,R^9m^4 + 278528\,R^{10}m^2 + 5599232\,R^9m^3 + 14548160\,R^8m^4 - 32768\,R^{10}m - 2076672\,R^9m^2 - 18182912\,R^8m^3 - 26604896\,R^7m^4 + 245760\,R^9m + 6653184\,R^8m^2 + 33337920\,R^7m^3 + 30093788\,R^6m^4 - 793600\,R^8m - 11929472\,R^7m^2 - 37735728\,R^6m^3 - 21560112\,R^5m^4 - 1024\,R^8 + 1449984\,R^7m + 13031296\,R^6m^2 + 26991516\,R^5m^3 + 9552844\,R^4m^4 + 6144\,R^7 - 1667840\,R^6m - 8798280\,R^5m^2 - 11907028\,R^4m^3 - 2392768\,R^3m^4 - 15488\,R^6 + 1292192\,R^5m +$

$3507263\,R^4m^2 + 2959404\,R^3m^3 + 259308\,R^2m^4 + 21440\,R^5 - 727060\,R^4m - 699902\,R^3m^2 - 316932\,R^2m^3 - 18116\,R^4 + 323380\,R^3m + 25823\,R^2m^2 + 10128\,R^3 - 113764\,R^2m + 8232\,Rm^2 - 4120\,R^2 + 26460\,Rm + 1232\,R - 2744\,m - 196.$

The following expression is for the factors of the discriminant of equation (2.7):

$$D_2 = 602112\,R^{10}m^4 - 745472\,R^{10}m^3 - 4856832\,R^9m^4 + 278528\,R^{10}m^2 + 5902336\,R^9m^3 + 17208512\,R^8m^4 - 32768\,R^{10}m - 2134016\,R^9m^2 - 20511488\,R^8m^3 - 34976928\,R^7m^4 + 245760\,R^9m + 7137536\,R^8m^2 + 40847552\,R^7m^3 + 44595420\,R^6m^4 - 807936\,R^8m - 13572992\,R^7m^2 - 50958800\,R^6m^3 - 36510768\,R^5m^4 - 1024\,R^8 + 1531904\,R^7m + 15975232\,R^6m^2 + 40752444\,R^5m^3 + 18735300\,R^4m^4 + 6144\,R^7 - 1868032\,R^6m - 11810232\,R^5m^2 - 20383092\,R^4m^3 - 5505408\,R^3m^4 - 15744\,R^6 + 1571232\,R^5m + 5244095\,R^4m^2 + 5822604\,R^3m^3 + 708588\,R^2m^4 + 22592\,R^5 - 981036\,R^4m - 1191294\,R^3m^2 - 726084\,R^2m^3 - 20292\,R^4 + 488332\,R^3m + 55647\,R^2m^2 + 12432\,R^3 - 191196\,R^2m + 17496\,Rm^2 - 5656\,R^2 + 49572\,Rm + 1872\,R - 5832\,m - 324.$$

The following expression is for the factors of equation (3.2):

$$F = (96\,m^4R^6 + 512\,m^4R^5 + 32\,m^4R^4\mu^4 + 76\,m^4R^4\mu^2 + 1108\,m^4R^4 + 128\,m^4R^3\mu^4 + 296\,m^4R^3\mu^2 + 1240\,m^4R^3 + 4\,m^4R^2\mu^6 + 193\,m^4R^2\mu^4 + 418\,m^4R^2\mu^2 + 749\,m^4R^2 + 8\,m^4R\mu^6 + 132\,m^4R\mu^4 + 256\,m^4R\mu^2 + 228\,m^4R - m^4\mu^8 + 30\,m^4\mu^4 + 56\,m^4\mu^2 + 27\,m^4 + 256\,M^3R^7 + 256\,M^3R^6\mu^2 + 2048\,m^3R^6 + 256\,m^3R^5\mu^4 + 1840\,m^3R^5\mu^2 + 6608\,m^3R^5 + 1344\,m^3R^4\mu^4 + 5456\,m^3R^4\mu^2 + 11376\,m^3R^4 + 48\,m^3R^3\mu^6 + 2824\,m^3R^3\mu^4 + 8400\,m^3R^3\mu^2 + 11352\,m^3R^3 + 144\,m^3R^2\mu^6 + 2988\,m^3R^2\mu^4 + 7064\,m^3R^2\mu^2 + 6556\,m^3R^2 - 16\,m^3R\mu^8 + 84\,m^3R\mu^6 + 1528\,m^3R\mu^4 + 3060\,m^3R\mu^2 + 2016\,m^3R - 16\,m^3\mu^8 - 12\,m^3\mu^6 + 276\,m^3\mu^4 + 524\,m^3\mu^2 + 252\,m^3 + 1024\,m^2R^7\mu^2 + 1024\,m^2R^7 + 512\,m^2R^6\mu^4 + 7168\,m^2R^6\mu^2 + 6656\,m^2R^6 + 3584\,m^2R^5\mu^4 + 22528\,m^2R^5\mu^2 + 18944\,m^2R^5 + 192\,m^2R^4\mu^6 + 10256\,m^2R^4\mu^4 + 40416\,m^2R^4\mu^2 + 30352\,m^2R^4 + 29312\,m^2R^3 + 768\,m^2R^3\mu^6 + 15488\,m^2R^3\mu^4 + 44032\,m^2R^3\mu^2 - 96\,m^2R^2\mu^8 + 816\,m^2R^2\mu^6 + 12704\,m^2R^2\mu^4 + 28720\,m^2R^2\mu^2 + 16928\,m^2R^2 - 192\,m^2R\mu^8 + 96\,m^2R\mu^6 + 5120\,m^2R\mu^4 + 10208\,m^2R\mu^2 + 5376\,m^2R - 96\,m^2\mu^8 - 144\,m^2\mu^6 + 720\,m^2\mu^4 + 1488\,m^2\mu^2 + 720\,m^2 + 1024\,mR^6\mu^4 + 2048\,mR^6\mu^2 + 1024\,mR^6 + 256\,mR^5\mu^6 + 6144\,mR^5\mu^4 + 11520\,mR^5\mu^2 + 5632\,mR^5 + 1280\,mR^4\mu^6 + 15424\,mR^4\mu^4 + 27008\,mR^4\mu^2 + 12864\,mR^4 - 256\,mR^3\mu^8 + 1728\,mR^3\mu^6 + 19840\,mR^3\mu^4 + 33472\,mR^3\mu^2 + 15616\,mR^3 - 768\,mR^2\mu^8 + 64\,mR^2\mu^6 + 13056\,mR^2\mu^4 + 22848\,mR^2\mu^2 + 10624\,mR^2 - 768\,mR\mu^8 - 1216\,mR\mu^6 + 3712\,mR\mu^4 + 8000\,mR\mu^2 + 3840\,mR - 256\,m\mu^8 - 576\,m\mu^6 + 192\,m\mu^4 + 1088\,m\mu^2 + 576\,m - 256\,R^4\mu^8 - 768\,R^4\mu^6 - 768\,R^4\mu^4 - 256\,R^4\mu^2 - 1024\,R^3\mu^8 - 3072\,R^3\mu^6 - 3072\,R^3\mu^4 - 1024\,R^3\mu^2 - 1536\,R^2\mu^8 - 4608\,R^2\mu^6 - 4608\,R^2\mu^4 - 1536\,R^2\mu^2 - 1024\,R\mu^8 - 3072\,R\mu^6 - 3072\,R\mu^4 - 1024\,R\mu^2 - 256\,\mu^8 - 768\,\mu^6 - 768\,\mu^4 - 256\,\mu^2).$$

Acknowledgements Both authors are very grateful to the anonymous referee for his/her deep review of the original version, valuable comments, and suggestions that helped us to improve notably this work.

References

[1] J. Bolyai and W. Bolyai, *Geometrische Untersuchungen*. Hrsg. P. Stäckel, Teubner, Leipzig-Berlin, 1913.

[2] R. Broucke, *Stability of periodic orbits in the elliptic, restricted three-body problem*, AIAA Journal, 7(1969), no. 6, 1003–1009.

[3] F. Diacu, *On the singularities of the curved n-body problem*. Trans. Amer. Math. Soc. **363**(2011), no. 4, 2249–2264. http://dx.doi.org/10.1090/S0002-9947-2010-05251-1

[4] ———, *Relative equilibria of the curved N-body problem*. Atlantis Press, Paris, 2012. http://dx.doi.org/10.2991/978-94-91216-68-8

[5] F. Diacu, R. Martínez, E. Pérez-Chavela, and C. Simó, *On the stability of tetrahedral relative equilibria in the positively curved 4-body problem*. Phys. D **256/257**(2013), 21–35. http://dx.doi.org/10.1016/j.physd.2013.04.007

[6] F. Diacu and E. Pérez-Chavela, *Homographic solutions of the curved 3-body problem*. J. Differential Equations **250**(2011), 340–366. http://dx.doi.org/10.1016/j.jde.2010.08.011

[7] F. Diacu, E. Pérez-Chavela, and M. Santoprete, *The n-body problem in spaces of constant curvature. Part I: Relative equilibria*. J. Nonlinear Sci. **22**(2012), 247–266. http://dx.doi.org/10.1007/s00332-011-9116-z

[8] ———, *The n-body problem in spaces of constant curvature. Part II: Singularities*. J. Nonlinear Sci. **22**(2012), 267–275. http://dx.doi.org/10.1007/s00332-011-9117-y

[9] L. C. García-Naranjo, J. C. Marrero, E. Pérez-Chavela, and M. Rodríguez-Olmos, *Classification and stability of relative equilibria for the two-body problem in the hyperbolic space of dimension 2*. J. Differential Equations **260**(2016), no. 7, 6375–6404. http://dx.doi.org/10.1016/j.jde.2015.12.044

[10] A. Hernández-Garduño and C. Stoica, *Lagrangian relative equilibria in a modified three-body problem with a rotationally symmetric ellipsoid*. SIAM J. Appl. Dyn. Syst. **14**(2015), no. 1, 221–252. http://dx.doi.org/10.1137/130949233

[11] W. Killing, *Die Mechanik in den nichteuklidischen Raumformen*. J. Reine Angew. Math. **98**(1885), 1–48. http://dx.doi.org/10.1515/crll.1885.98.1

[12] R. Martínez and C. Simó, *On the stability of the Lagrangian homographic solutions in a curved three-body problem on* S^2. Discrete Cont. Dyn. Syst. **33**(2013), 1157–1175.

[13] J. F. Cariñena, M. F. Rañada, and M. Santander, *Central potentials on spaces of constant curvature: the Kepler problem on the two-dimensional sphere* S^2 *and the hyperbolic plane* H^2. J. Math. Phys. **46**(2005), 052702, 25. http://dx.doi.org/10.1063/1.1893214

[14] R. Lipschitz, *Extension of the planet-problem to a space of n dimensions and constant integral curvature*. Quart. J. Pure Appl. Math. **12**(1873), 349–370.

[15] N. I. Lovachevski, *The new foundations of geometry with full theory of parallels.*(Russian) In: Collected Works, V. 2, GITTL, Moscow, 159, 1949, pp. 1835–1838.

[16] R. A. Mardling, *Resonance, chaos and stability in the general three-body problem*. Dynamical Evolution of Dense Stellar Systems Proceedings IAU Symposium, 246, 2007.

[17] E. Pérez-Chavela and J. M. Sánchez Cerritos, *Stability of Euler-Type relative equilibria in the curved three body problem*. In: Hamiltonian systems and celestial mechanics, Trends in Mathematics, 4, Springer, Switzerland, 2015, pp. 59–63. http://dx.doi.org/10.1007/978-3-319-22129-8_11

[18] E. Pérez-Chavela and J. G. Reyes-Victoria, *An intrinsic approach in the curved n-body problem. The positive curvature case*. Trans. Amer. Math. Soc. **364**(2012), no. 7. 3805–3827. http://dx.doi.org/10.1090/S0002-9947-2012-05563-2

[19] A. B. Schepetilov, *Nonintegrability of the two-body problem in constant curvature spaces*. J. Phys. A. Gen. **39**(2006), 5787–5806. http://dx.doi.org/10.1088/0305-4470/39/20/011

[20] E. Schering, *Die Schwerkraft im Gaussischen Raume*. Nachrichten von der Königl. Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen **1870**(1870), 311–321.

[21] S. Zhu, *Eulerian Relative Equilibria of the curved three 3-body problem in* S^2. Proc.s Amer. Math. Soc. **142**(2014), 2837–2848. http://dx.doi.org/10.1090/S0002-9939-2014-11995-2

Departamento de Matemáticas, Instituto Tecnológico Autónomo de México, México D.F., México
e-mail: ernesto.perez@itam.mx

Departamento de Matemáticas, Universidad Autónoma Metropolitana - Iztapalapa, México D.F., México
e-mail: sanchezj01@gmail.com