Canad. Math. Bull. Vol. 18 (4), 1975

A BANACH ALGEBRA STRUCTURE FOR H^p

BY
NEIL M. WIGLEY

The Hardy space $H^p = H^p(U)$, where $1 \le p \le \infty$ and U is the unit disc $|z|<1$, is shown to be an algebra under the product

$$(f * g)(z) = \frac{d}{dz}\int_0^z f(z-t)g(t)\,dt, \qquad f, g \in H^p.$$

Moreover for each p, $1 \le p \le \infty$, there exists a constant C_p such that $\|f*g\|_p \le C_p\,\|f\|_p\,\|g\|_p$. The above product is also known to be commutative and associative and has the identity $f(z)\equiv 1$. Thus by embedding H^p into $\mathscr{B}(H^p)$, the Banach algebra of bounded linear operators on H^p, with the mapping

$$f \to T_f,\ T_f(g) = f * g, \quad \text{for} \quad f, g \in H^p,$$

the Banach space H^p becomes a commutative Banach algebra with identity. An operator T_f is shown to be invertible if and only if $f(0)\ne 0$. It follows that there is only one maximal ideal, the set of functions which vanish at the origin, and the spectrum of each T_f is the singleton $\{f(0)\}$.

1. **Introduction.** We shall use the notation of Duren [1]. U is the open unit disc of the complex plane, $H(U)$ is the space of functions holomorphic on U and $H^p = H^p(U)$ $(1 \le p < \infty)$ is the space of functions $f \in H(U)$ with finite integral means

$$\|f\|_p = \lim_{r\to 1} M_p(r,f) < +\infty$$

where

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\,d\theta\right)^{1/p}.$$

H^∞ is the space of bounded analytic functions on U with the supremum norm. We define a product on functions in $H(U)$:

$$(f * g)(z) = \frac{d}{dz}\int_0^z f(z-t)g(t)\,dt = \int_0^z f'(z-t)g(t)\,dt + f(0)g(z).$$

This product is known to be commutative and associative and has no zero divisors [3]. The function $f(z)\equiv 1$ is the identity.

Received by the editors October 31, 1973 and, in revised form, December 11, 1974.

This research was supported by the National Research Council and the Summer Research Institute of the Canadian Mathematical Congress. The author wishes to thank the referee for his helpful suggestions.

https://doi.org/10.4153/CMB-1975-106-4 Published online by Cambridge University Press

$H(U)$ clearly becomes an algebra with the above product. If X is a linear subspace of $H(U)$ which is multiplicatively closed under the above product, we shall denote the resulting algebra by $(X, *)$. The algebra $(H, *)$ can be embedded into the formal power series ring $\mathbb{C}$ $[[Z]]$ as follows. If $f(z)=z^n$ and $g(z)=z^m$ an easy induction shows that

$$(f*g)(z) = \frac{m!\,n!}{(m+n)!}\,z^{m+n}.$$

Let $f \in H(U)$ have the Taylor series $\sum a_n z^n/n!$ and define $B\colon H(U) \to \mathbb{C}[[Z]]$ by

$$(Bf)(Z) = \sum a_n Z^n.$$

In general this series does not converge; in fact it has a positive radius of convergence ρ if and only if f is an entire function of finite type $1/\rho$ ([2], p. 73). If $g(z)=\sum b_m z^m/m!$ then

$$(f*g)(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{a_k b_{n-k}}{n!}\,z^n,$$

the series converging because it is majorized by the usual Cauchy product series representing $f(z)g(z)$. It follows that $B(f*g)(Z)=B(f)(Z)B(g)(Z)$, the product on the right being the usual Cauchy product in $\mathbb{C}[[Z]]$. Thus B is an algebra isomorphism of $H(U)$ onto a subalgebra of $\mathbb{C}[[Z]]$.

With a different product, though also of convolution type, H^p has another Banach algebra structure which is very different from the one presented here; see, for instance [5].

2. **Main Theorems.** We now turn to the H^p spaces and state a theorem which will be proved later.

THEOREM 1. *Let* $1 \le p \le \infty$ *and let* $f, g \in H^p$. *Then* $f*g \in H^p$ *and there exists a constant* C_p *depending only on p such that*

$$\|f*g\|_p \le C_p\,\|f\|_p\,\|g\|_p.$$

Moreover given $f \in H^p$ *there exists* $g \in H^p$ *such that* $(f*g)(z)\equiv 1$ *if and only if* $f(0)\neq 0$.

It follows that H^p becomes a Banach algebra if we identify elements $f \in H^p$ with $T_f \in \mathscr{B}(H^p)$. For details see [4], p. 860. Each operator $T_f \neq 0$ is $1-1$ because $(H(U), *)$ is an integral domain. T_f is invertible if and only if f is invertible in $(H^p, *)$ and this happens if and only if $f(0)\neq 0$. It follows that the operators T_f with $f(0)=0$ form a maximal ideal and this is the only maximal ideal since it is exactly the set of singular elements.

In the algebra $(H^p, *)$ there are many ideals which are not maximal. If $f(0)=0$ it follows quickly from the definition of the $*$-product that $(f*g)'=f'*g$ for $f, g \in H(U)$. Let $1\le q\le p\le\infty$ and consider the set $I_q^p=\{f \in H^p : f(0)=0 \text{ and } f' \in H^q\}$.

We claim I_q^p is an ideal in H^p. For let $f \in I_q^p$ and $g \in H^p$. Then $(f*g)(0)=0$ and $(f*g)'=f'*g$. Since $f' \in H^q$ and $g \in H^p \subseteq H^q$ we get $(f*g)' \in H^q$ and thus I_q^p is an ideal in H^p. It is not closed as the following example shows. Let $f_n(z)=(1-z)^{(1/n)-(1/q)+1}$, $n=1, 2, \ldots$. It is easy to show that $f_n \in H^p$, $f_n' \in H^q$, $\lim f_n = f \in H^p$, yet $f' \notin H^q$. If we set $g_n = f_n - 1$ then $g_n \in I_q^p$ yet $g = \lim g_n \notin I_q^p$. Other ideals can be gotten by considering functions f for which $f(0)=f'(0)=0$ and $f'' \in H^q$, etc.

Let $1 \leq p < \infty$ and $f \in H^p$. Then $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ exists almost everywhere and is in $L^p(T)$. Let

$$\omega_p(t, f^*) = \sup_{0<h\leq t} \left\{ \int_0^{2\pi} |f^*(e^{i(\theta+h)}) - f^*(e^{i\theta})|^p \, d\theta \right\}^{1/p}.$$

Let $H^p\Lambda_1^p$ be the set of $f \in H^p$ for which $(1/t)\omega_p(t, f^*)$ is bounded as $t \to 0$. A theorem of Hardy and Littlewood states that $f \in H^p\Lambda_1^p$ if and only if $f' \in H^p$. In addition $f' \in H^1$ if and only if $f \in C(\bar{U})$ and f^* is absolutely continuous, so $H^p\Lambda_1^p \subseteq C(\bar{U})$. Let $H\Lambda_1$ denote the set of $f \in C(\bar{U})$ for which f^* satisfies a Lipschitz condition, i.e. there exists K such that $|f^*(e^{i\theta}) - f^*(e^{it})| \leq K|\theta - t|$. Then $f \in H\Lambda_1$ if and only if $f' \in H^\infty$. From these facts we get the following theorems.

THEOREM 2. *Let $1 \leq p < \infty$, $f \in H^p$, $f' \in H^p$ and $f(0)=0$. Then T_f maps H^p into the subspace of $H^p\Lambda_1^p$ of functions which vanish at the origin. Moreover T_f is onto if and only if $f'(0) \neq 0$.*

Proof. Let $g \in H^p$. Since f belongs to the ideal I_p^p of H^p, so does $T_f(g) = f*g$. Let $f'(0) \neq 0$ and $h \in H^p\Lambda_1^p$, $h(0)=0$. Let g be the function in H^p such that $f'*g = h'$. Then $f*g=h$ so is T_f onto. If $f'(0)=0$ then $f*g$ has a zero of order at least two at the origin, so T_f cannot be onto.

THEOREM 3. *Let $f \in H^\infty$, $f' \in H^\infty$ and $f(0)=0$. Then T_f maps H^∞ into the subspace of $H\Lambda_1$ of functions which vanish at the origin. Moreover T_f is onto if and only if $f'(0) \neq 0$.*

Proof. Same as for theorem 2 with $p = \infty$.

3. **Proof of Theorem 1.** We begin with some lemmas:

LEMMA 1. *If $f \in H^p$ and $|z| \leq \frac{1}{2}$ then $|f(z)| \leq 2^{2/p} \|f\|_p$. If in addition $f(0)=0$ then $|f(z)| \leq 2^{(2/p)+1} |z| \, \|f\|_p$.*

Proof. From [1] p. 36 we get

$$|f(z)| \leq 2^{1/p} \|f\|_p (1-r)^{-1/p}$$

for $f \in H^p$ and $z \in U$. The first part of the lemma follows immediately. For the second part observe that $f(z)/z$ is bounded and analytic for $|z| < \frac{1}{2}$. Its supremum is taken on the circle $|z| = \frac{1}{2}$.

LEMMA 2. *Let $f \in H(U)$ and $r<\rho<1$. Then*

$$M_p(r, f') \le \frac{M_p(\rho, f)}{\rho - r}$$

Proof. With $|z|=r$ and $|\zeta|=\rho$ we have

$$f'(z) = \frac{1}{2\pi i}\int_{|\zeta|=\rho} \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta.$$

Then

$$|f'(z)| \le \frac{1}{2\pi}\int_0^{2\pi} \frac{|f(\zeta)|\,\rho\,d\phi}{\rho^2+r^2-2\rho r\cos(\theta-\phi)}$$

$$\le \frac{1}{2\pi(\rho-r)}\int_0^{2\pi} |f(\rho e^{i\theta})|\,\frac{\rho^2-r^2}{\rho^2+r^2-2\rho r\cos(\theta-\phi)}\,d\phi$$

$$= \frac{1}{2\pi(\rho-r)}\int_0^{2\pi} |f(\rho e^{i(t+\theta)}|\,\frac{\rho^2-r^2}{\rho^2+r^2-2\rho r\cos t}\,dt.$$

An application of Jensen's inequality with respect to the measure $(\rho^2-r^2)\,dt/2\pi(\rho^2+r^2-2\rho r \text{ cost})$ yields

$$M_p(r, f') \le \frac{M_p(\rho, f)}{\rho-r}\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{\rho^2-r^2}{\rho^2+r^2-2\rho r\cos t}\,dt\right)^{1/p} = \frac{M_p(\rho, f)}{\rho-r}$$

LEMMA 3. *Let $h \in H^p$ have a zero at the origin of order at least n. Then there exists $g \in H^p$ such that $h(z)=z^n g(z)$, $\|h\|_p=\|g\|_p$ and $M_p(r, h)=r^n M_p(r, g)$.*

Proof. This is clear from the definitions.

Proof of Theorem 1. We first consider the case $p=\infty$. Let $f, g \in H^\infty$. We must show $f * g \in H^\infty$. For the moment assume that $f(0)=g(0)=0$. Then

$$(f*g)(z) = \int_0^z f'(z-t)g(t)\,dt = \int_0^z f(z-t)g'(t)\,dt = \int_0^r f((r-\tau)e^{i\theta})g'(\tau e^{i\theta})e^{i\theta}\,d\tau.$$

By Schwarz's lemma $|f(z-t)| \le \|f\|_\infty (r-\tau)$ and by Cauchy's estimate $|g'(t)| \le \|g\|_\infty (1-\tau)^{-1}$ where $\tau=|t|$. Hence

$$|(f*g)(z)| \le \|f\|_\infty\,\|g\|_\infty \int_0^r \frac{r-\tau}{1-\tau}\,d\tau \le \|f\|_\infty\,\|g\|_\infty$$

so $\|f*g\|_\infty \le \|f\|_\infty\,\|f\|_\infty$.

For arbitrary f, $g \in H^\infty$ set $f(z)=\hat{f}(z)+f(0)$, $g(z)=\hat{g}(z)+g(0)$. Then

$$(f*g)(z) = (\hat{f}*\hat{g})(z)+f(0)\hat{g}(z)+g(0)f(z)$$

https://doi.org/10.4153/CMB-1975-106-4 Published online by Cambridge University Press

and

$$\|f*g\|_\infty \le \|\hat{f}\|_\infty \|\hat{g}\|_\infty + \|f\|_\infty \|\hat{g}\|_\infty + \|g\|_\infty \|f\|_\infty \le 7\,\|f\|_\infty \|g\|_\infty.$$

Assume next that $f\in H^\infty$ and $f(0)=0$. We shall show there exists $g\in H^\infty$ such that $g*(1-f)=1$. From the power series of $f*f$ we see that $f*f$ has a zero of order ≥ 2 at the origin. Thus $(f*f)(z)/z^2$ is analytic and bounded for $|z|<1$ and

$$\|(f*f)(z)/z^2\|_\infty = \|f*f\|_\infty \le \|f\|_\infty^2$$

Hence

$$|(f*f)(z)| \le r^2\,\|f\|_\infty^2.$$

Assume for $n\geq 2$ that

$$|f^{[n]}(z)| \le \frac{r^n\,\|f\|_\infty^n}{(n-1)!},$$

where $f^{[n]}$ denotes the n-fold $*$-product of f with itself. This is indeed the case for $n=2$. Then

$$\begin{aligned}|f^{[n+1]}(z)| &= \left|\int_0^z f^{[n]}(z-t)f'(t)\,dt\right| \\ &\le \frac{\|f\|_\infty^{n+1}}{(n-1)!}\int_0^r \frac{(r-\tau)^n}{1-\tau}\,d\tau \le \frac{\|f\|_\infty^{n+1}}{(n-1)!}\int_0^r (1-\tau)^{n-1}\,d\tau \le \frac{\|f\|_\infty^{n+1}}{n!}.\end{aligned}$$

But $f^{[n+1]}(z)$ has a zero of order $\geq n+1$ at the origin, and hence

$$|f^{[n+1]}(z)| \le r^{n+1}\,\|f^{[n+1]}\|_\infty \le r^{n+1}\frac{\|f\|_\infty^{n+1}}{n!},$$

which completes the inductive step. Consequently, the series

$$g(z) = \sum_{n=0}^\infty f^{[n]}(z)$$

is majorized by the series

$$1+\sum_{n=1}^\infty \frac{r^n\,\|f\|_\infty^n}{(n-1)!}$$

which is convergent for any r. Thus $g\in H^\infty$ and $g*(1-f)=1$.

We now consider the case $1\leq p<\infty$. Let $f, g\in H^p$. We must show that $f*g\in H^p$. As in the H^∞ case it is sufficient to assume $f(0)=g(0)$ and to show $\|f*g\|_p\leq C_p\,\|f\|_p\,\|g\|_p$ where C_p depends only on p. Then we have

$$\begin{aligned}(f*g)(z) &= \int_0^z f(z-t)g'(t)\,dt \\ &= \int_0^{1/2} f((r-\tau)e^{i\theta})g'(\tau e^{i\theta})e^{i\theta}\,d\tau + \int_{1/2}^r f((r-\tau)e^{i\theta})g'(\tau e^{i\theta})e^{i\theta}\,d\tau\end{aligned}$$

for $r>\frac{1}{2}$. To estimate the derivative $g'(\tau e^{i\theta})$ in the first integral on the right we use

Cauchy's integral theorem followed by the integral form of Minkowski's inequality:

$$g'(\tau e^{i\theta}) = \frac{1}{2\pi i}\int_{|\xi|=3/4}\frac{g(\zeta)}{(\zeta-\tau e^{i\theta})^2}\,d\zeta$$

$$|g'(\tau e^{i\theta})| \le \frac{6}{\pi}\int_0^{2\pi}|g(\tfrac{3}{4}e^{i\mu})|\,d\mu$$

$$|g'(\tau e^{i\theta})| \le 12\left(\int_0^{2\pi}|g(\tfrac{3}{4}e^{i\mu})|^p\frac{1}{2\pi}\,d\mu\right)^{1/p} \le 12\,\|g\|_p,$$

which is valid for $\tau<\frac{1}{2}$. Then from lemma 1 we obtain

$$|(f*g)(z)| \le 12\,\|g\|_p\int_0^{1/2}|f((r-\tau)e^{i\theta})|\,d\tau+2^{(2/p)+1}\,\|f\|_p\int_{1/2}^{r}(r-\tau)^{(2/p)+1}\,|g'(\tau e^{i\theta})|\,d\tau$$

Using Minkowski's inequality again and lemma 2 we get

$$M_p(f*g,r) \le 12\,\|f\|_p\,\|g\|_p$$
$$+2^{(2/p)+1}\,\|f\|_p\int_{1/2}^{r}(r-\tau)^{(2/p)+1}M_p(r,g)(r-\tau)^{-1}\,d\tau \le C_p\,\|f\|_p\,\|g\|_p$$

Assume now that $f\in H^p$ and $f(0)=0$. We shall show as in the H^∞ case that the series $\sum f^{[n]}$ converges and belongs to H^p. Assume for $n\ge 2$ that

$$\|f^{[n]}\|_p \le \frac{C_p}{(n-1)!}\,\|f\|_p^n.$$

This is true for $n=2$. We write $f^{[n]}(z)=z^n g(z)$ as in lemma 3 and obtain through lemma 2 the inequality

$$|f^{[n+1]}(z)| = \left|\int_0^r f^{[n]}((r-\tau)e^{i\theta})f'(\tau e^{i\theta})\,d\tau\right| \le \int_0^r (r-\tau)^{n-1}|g((r-\tau)e^{i\theta}|\,M_p(\tau,f)\,d\tau$$

An application of Minkowski's inequality yields

$$M_p(r,f^{[n+1]}) \le \int_0^r (r-\tau)^{n-1}M_p(r-\tau,q)M_p(\tau,f)\,d\tau$$
$$\le \|g\|_p\,\|f\|_p\int_0^r(r-\tau)^{n-1}\,d\tau = \frac{r^n}{n}\,\|g\|_p\,\|f\|_p$$

Letting $r\to 1$ and using the inductive hypothesis we find

$$\|f^{[n+1]}\| \le \frac{1}{n}\,\|f^{[n]}\|_p\,\|f\|_p \le \frac{C_p}{n!}\,\|f\|_p^{n+1}$$

We conclude that the series $\sum f^{[n]}$ is majorized in the H^p-norm by the series

$$1+\|f\|_p+C_p\sum_{n=2}^{\infty}\frac{\|f\|_p^n}{(n-1)!}$$

which converges for any $\|f\|_p<+\infty$. This completes the proof.

BIBLIOGRAPHY

1. P. L. Duren, *Theory of H^p Spaces*, Academic, New York.
2. R. P. Boas, Jr., *Entire Functions*, Academic, New York.
3. Ditkin and Prudnikov, *Integral Transforms and Operational Calculus*, Pergamon, New York.
4. Dunford and Schwartz, *Linear Operators*, Interscience, New York.
5. P. Porcelli, *Linear Spaces of Analytic Functions*, Rand McNally, Chicago.