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SIMILARITY INVARIANT SEMIGROUPS GENERATED BY NON-FREDHOLM OPERATORS

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Let $A \in B(H)$ be a bounded non-compact operator that is not semi-Fredholm. The similarity invariant semigroup generated by A is shown to consist of all operators that are not semi-Fredholm and satisfy obvious inequalities for the nullity and co-nullity.

1. INTRODUCTION

Let A be a bounded operator on a separable, infinite dimensional Hilbert space H. Moreover, assume that A is not in the set $\mathbb{C} + K(H)$ of operators expressible as a sum of a scalar multiple of identity and a compact operator. What is the smallest similarity invariant semigroup containing operator A? Equivalently, which operators can be expressed as products of operators, similar to A?

A partial answer to the above question was obtained in 2003 by Fong and Sourour [**6**]. They proved that if operator $A \notin \mathbb{C} + K(H)$ is invertible, every invertible operator is a product of operators, similar to A. The author extended their results to semi-invertible operators [**7**] and later to semi-Fredholm operators in a so far unpublished article. In these cases we must account for the Fredholm index, which makes precisely specifying the semigroup very difficult; we only manage to prove that it contains all operators with index that is a large enough multiple of $\operatorname{ind} A$.

In this article we consider the operators that are not semi-Fredholm, termed *non-Fredholm* operators. We shall see that, although harder to prove, the results are more conclusive than in the case of semi-Fredholm operators.

Throughout this article we assume that H is a separable, infinite dimensional Hilbert space. All operators appearing in the article are bounded.

We shall denote the null-space of the operator A by $\ker A$ and its range by $\operatorname{ran} A$. The *nullity* of an operator is defined as

$$\operatorname{nul} A = \dim(\ker A).$$

The *co-nullity* of A is the nullity of A^* and equals $\dim(\operatorname{ran} A)^{\perp}$. An operator is semi-Fredholm if it has closed range and at least one of the nullities $\operatorname{nul} A$, $\operatorname{nul} A^*$ is finite.

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Conversely, operator A is non-Fredholm if and only if either $\operatorname{nul} A$ and $\operatorname{nul} A^*$ are both infinite or $\operatorname{ran} A$ is non-closed.

Notation $A \sim B$ stands for "operator A is similar to B". By $\mathcal{S}(A)$ we denote the *similarity invariant semigroup generated by A*. It is defined to be the smallest semigroup of operators, invariant for similarity, that contains A. The equivalent definition is that $\mathcal{S}(A)$ is the semigroup, generated by the similarity orbit of A.

To express our results in a compact form it is useful to define two simple relations on the set $\mathbb{N} \cup \{0, \infty\}$ of all possible nullities of an operator. For a pair $n, m \in \mathbb{N} \cup \{0, \infty\}$ we write $n \approx m$ if and only if n and m are both zero, both nonzero finite or both infinite. We write $n \gtrapprox m$ if and only if $n \approx m$ and $n \geqslant m$. Using this notation, the main theorem of this article can be stated as follows.

THEOREM 1. *Let $A \in B(H)$ be a non-compact non-Fredholm operator and assume $\operatorname{nul} A \approx \operatorname{nul} A^*$. The semigroup $\mathcal{S}(A)$ equals the set of all non-Fredholm operators $X \in B(H)$ for which the conditions $\operatorname{nul} X \gtrapprox \operatorname{nul} A$ and $\operatorname{nul} X^* \gtrapprox \operatorname{nul} A^*$ hold.*

The need for the assumption $\operatorname{nul} A \approx \operatorname{nul} A^*$ will be discussed at the end of the article. It is of purely technical nature and we conjecture that it can be dropped.

2. Operators with infinite nullities and co-nullities

It turns out that of all the non-Fredholm operators, the easiest to work with are the operators for which the equality $\operatorname{nul} A = \operatorname{nul} A^* = \infty$ holds. The reason for this is the following theorem [**2**, Lemma 3.3].

THEOREM. (Dawlings) *Let T be a bounded operator on a separable Hilbert space and assume $\operatorname{nul} T = \operatorname{nul} T^* = \infty$. Then T is a product of 3 bounded idempotents with infinite dimensional null-spaces.*

We shall require a slightly stronger version of the above statement.

LEMMA 1. *Let $P \in B(H)$ be a bounded idempotent with infinite nullity and rank. Every operator $T \in B(H)$ satisfying $\operatorname{nul} T = \operatorname{nul} T^* = \infty$ is a product of 3 operators similar to P.*

PROOF: By the theorem of Dawlings, T is a product of 3 bounded idempotents with infinite dimensional null-spaces. Since bounded idempotents are similar if and only if they have isomorphic null-spaces and ranges, all we have to prove is that the idempotents can be chosen in such a way that they all have infinite ranks.

If at least one of the idempotents has finite rank, then so does T. It is easy to see that in this case H can be split as $H = M \oplus N$ where M and N are infinite dimensional, so that $T = T_1 \oplus 0$ according to this decomposition. Operator $T_1 \in B(M)$ satisfies the requirements of the above theorem, therefore $T_1 = P_1 P_2 P_3$ where P_i are idempotents with infinite nullities on M. Let Q_1, Q_2 and Q_3 be any mutually orthogonal projectors with infinite ranks on the space N. Then

$$T_1 \oplus 0 = (P_1 \oplus Q_1)(P_2 \oplus Q_2)(P_3 \oplus Q_3),$$

which is a product of the required form. $\square$

THEOREM 2. *Let $A \in B(H)$ be a non-compact operator that satisfies the equality $\operatorname{nul} A = \operatorname{nul} A^* = \infty$. Every operator $B \in B(H)$ with infinite nullity and co-nullity is a product of 12 operators similar to A.*

PROOF: Since A is neither compact nor semi-Fredholm, $A \notin \mathbb{C} + K(H)$, [**7**, Proposition 1] states the following for any $A \notin \mathbb{C} + K(H)$. For every unitary operator U there exists a product of two operators similar to A which has the form $\alpha U \oplus Y$ where $\alpha > 0$ is a number and Y is a bounded operator. Here we choose $U = I$ hence the product of two operators similar to A has the form $\alpha I \oplus Y$. It is obvious that operator Y satisfies the condition $\operatorname{nul} Y = \operatorname{nul} Y^* = \infty$.

There exist closed subspaces $M \subseteq \ker Y$, $N \subseteq \ker Y^*$ such that $\dim M = \dim M^\perp = \infty$ and $\dim N = \dim N^\perp = \infty$. Let W be a unitary operator that maps N onto $M^\perp$ and $N^\perp$ onto M. Since $\operatorname{ran} Y \subseteq N^\perp$, we have $\operatorname{ran} WY \subseteq M$ and $YWYW^* = 0$. A product of two operators similar to $\alpha I \oplus Y$ is

$$(\alpha I \oplus Y)(I \oplus W)(\alpha I \oplus Y)(I \oplus W)^* = \alpha^2 (I \oplus 0).$$

Since $I \oplus 0$ is an idempotent with infinite nullity and rank we can use Lemma 1 to show that B is a product of 3 operators similar to $\alpha(I \oplus 0)$. Consequently, B is a product of 12 operators similar to A. $\square$

The following characterisation shows that in the Calkin algebra every element without both left and right inverse behaves like a Hilbert space operator with infinite nullity and co-nullity.

LEMMA 2. *Let A be a bounded operator on a Hilbert space H. The following statements are equivalent.*

(i) *A is not semi-Fredholm.*

(ii) *The image of A in the Calkin algebra is neither left nor right invertible.*

(iii) *There exist orthogonal projectors P, Q with infinite ranks on H such that $PA \in K(H)$ and $AQ \in K(H)$*

(iv) *There exists $K \in K(H)$ such that $\operatorname{nul}(A - K) = \operatorname{nul}(A - K)^* = \infty$.*

PROOF: The first two statements are equivalent by the definition; for the proof that (iii) is equivalent to (ii) see [**4**, Theorem 1.1].

That (iv) implies (i) is a consequence of the compact perturbation theorem for semi-Fredholm operators. To prove the opposite, let P and Q be the projectors from (iii).

$$\begin{aligned} A &= \big(P + (I - P)\big)A\big(Q + (I - Q)\big) \\ &= (I - P)A(I - Q) + PA(I - Q) + (I - P)AQ + PAQ \end{aligned}$$

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Operator

$$K = PA(I - Q) + (I - P)AQ + PAQ$$

is compact by (iii). The statement (iv) now follows since

$$A - K = (I - P)A(I - Q). \qquad \square$$

PROPOSITION 1. *Let A and B be bounded non-Fredholm operators. If the operator A is not compact, there exists $K \in K(H)$ such that $B + K$ is a product of* 12 *operators similar to A.*

PROOF: By statement (iv) of the preceding lemma and Theorem 2 there exist compact operators K', $K'' \in K(H)$ such that $B - K''$ is a product of 12 operators similar to $A - K'$. From the fact that $K(H)$ is an ideal it follows that B is up to a compact perturbation a product of 12 operators similar to A. $\square$

COROLLARY 1. *The semigroup of all elements of the Calkin algebra that have neither left nor right inverse has no non-trivial subsemigroup, invariant for similarity.*

3. INVARIANT SEMIGROUPS OF DENSE OPERATORS

Operator $A \in B(H)$ is *dense* if it is injective and its range is dense in H but not equal to H. Equivalently, A is dense if and only if both A and A^* are injective but not surjective.

It is easy to show that the set of all dense operators is a similarity invariant semigroup: if A and B are dense, operators AB and $(AB)^* = B^*A^*$ are both injective but not surjective. Moreover, for every invertible operator T, operators TAT^{-1} and its adjoint $T^{-1*}A^*T^*$ are injective but not surjective.

To study similarity invariant semigroups of dense operators we shall need some basic results from the theory of operator ranges. See the survey article [5] for more information on this topic.

Bounded operators A and B on a Hilbert space are *equivalent* if there exist invertible operators X and Y such that $B = YAX$. Denote by $\operatorname{ran} A$ and $\operatorname{ran} B$ ranges of A and B. We say that $\operatorname{ran} A$ and $\operatorname{ran} B$ are *similar* if $\operatorname{ran} A = T(\operatorname{ran} B)$ for some invertible operator T. The following proposition [3] characterises equivalence of operators in terms of operator ranges.

PROPOSITION 2. *Let A and B be bounded operators on a Hilbert space.*

(i) *There exists an operator $X \in B(H)$ that solves the operator equation $B = AX$ if and only if $\operatorname{ran} B \subseteq \operatorname{ran} A$. Moreover, operator X is unique if we further require that $\ker X = \ker B$ and $\ker X^* = \ker A$.*

(ii) *There exists an invertible operator X satisfying $B = AX$ if and only if $\operatorname{ran} A = \operatorname{ran} B$ and $\operatorname{nul} A = \operatorname{nul} B$.*

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(iii) *A and B are equivalent if and only if their ranges are similar and* $\operatorname{nul} A = \operatorname{nul} B$.

Not every linear subspace of a Hilbert space is an operator range; the characterisation below is a part of [**5**, Theorem 1.1].

Proposition 3. *Linear subspace $\mathcal{R} \subseteq H$ is an operator range if and only if there exists a sequence $\{H_n\}_{n\in\mathbf{N}}$ of mutually orthogonal subspaces of H such that*

$$\mathcal{R} = \left\{ \sum_{n=1}^{\infty} x_n\,;\ x_n \in H_n \text{ and } \sum_{n=1}^{\infty} (2^n \|x_n\|)^2 < \infty \right\}.$$

In this case, $\overline{\mathcal{R}} = \sum_{n=1}^{\infty} H_n$.

It is obvious that operator ranges $\mathcal{R}$ and $\mathcal{Q}$ are similar if and only if there exist sequences $\{H_n\}_{n\in\mathbf{N}}$ and $\{K_n\}_{n\in\mathbf{N}}$ that belong to $\mathcal{R}$ and $\mathcal{Q}$ respectively in the sense of the above proposition such that

(i) $\dim H_n = \dim K_n$ for all $n \in \mathbb{N}$,

(ii) $\dim \left(\sum_{n=1}^{\infty} H_n \right)^{\perp} = \dim \left(\sum_{n=1}^{\infty} K_n \right)^{\perp}$.

([**5**, Theorem 3.3] states a condition for *any* two sequences of orthogonal subspaces to belong to similar operator ranges. We don't need it in all its generality.)

Lemma 3. *Let $X, Y \in B(H)$ be dense operators on H. Assume that for every dense non-compact operator Z the semigroup $\mathcal{S}(Z)$ contains some operators X' and Y' equivalent to X and Y respectively. Then for every dense non-compact operator A the semigroup $\mathcal{S}(A)$ contains operator $X \oplus Y$.*

Proof: Let U be the bilateral shift of infinite multiplicity. Since A is not in the set $\mathbb{C} + K(H)$, by [**7**, Proposition 1] there exists a product Z of two operators similar to A that has the form $\alpha U \oplus B$ for some $\alpha > 0$ and some bounded operator B.

It is obvious that the operator Z is dense and non-compact. Moreover, Z is similar to $\alpha U \oplus Z$ because of the similarity $\alpha U \sim \alpha U \oplus \alpha U$.

By the assumption, $\mathcal{S}(Z)$ contains an operator X' equivalent to X. Then $X' = P_1 X P_2$ where P_1 and P_2 are invertible operators. The product $P_1 X P_2$ is obviously similar to $P_2 P_1 X$. Setting $P = P_2 P_1$ we see that $PX \in \mathcal{S}(Z)$ for some invertible operator P. By analogy there exists an invertible operator Q such that $YQ \in \mathcal{S}(Z)$.

Then [**7**, Lemma 1] states that every invertible operator is a product of 6 (or more) operators similar to bilateral shift of infinite multiplicity. In particular, P^{-1} and Q^{-1} can both be expressed as products of operators similar to αU. Thus $PX \oplus Q^{-1}$ and $P^{-1} \oplus YQ$ can both be expressed as products of operators similar to $\alpha U \oplus Z$. Their product

$$(P^{-1} \oplus YQ)(PX \oplus Q^{-1}) = X \oplus Y$$

is therefore an element of $\mathcal{S}(A)$.

NOTE. there might be a problem if X' is a product of less than 6 operators, similar to Z. In that case we define $\widetilde{Z}$ to be $Z^k \sim \alpha^k U \oplus B^k$ for sufficiently large k and use the fact that $X' \in \mathcal{S}(\widetilde{Z})$. ∎

An operator range is of type J_S if all the spaces H_n in Proposition 3 are infinite dimensional and their sum equals H. We define a canonical operator with range of type J_S to be $J_0 = \bigoplus_{n=1}^{\infty} 2^{-n} I_{H_n}$. Operator J_0 is obviously dense.

LEMMA 4. *Let A be a non-compact dense operator. The semigroup $\mathcal{S}(A)$ contains operator J_0.*

PROOF: Let Z be any non-compact, dense operator. By Proposition 1 there exists an operator $K \in K(H)$ such that $T = J_0 + K$ is a product of 12 operators similar to Z.

Operators T and $\sqrt{TT^*}$ have the same range. Because J_0 is positive and it equals T in the Calkin algebra, we have $\sqrt{TT^*} = J_0 + K'$ for some $K' \in K(H)$. For each $n \in \mathbb{N}$, 2^{-n} is an eigenvalue of infinite multiplicity of J_0 and that doesn't change if we perturb it by a compact operator. Therefore $\sqrt{TT^*}$ is positive and has 2^{-n} as an eigenvalue of infinite multiplicity for all $n \in \mathbb{N}$.

Now we use the fact that for a positive operator we can use images of spectral projections $E[2^{-n}, 2^{-n+1})$ as spaces H_n in Proposition 3. They are all infinite dimensional, therefore T has operator range of type J_S. Operators T and J_0 are both dense and have similar ranges. By Proposition 2 they are equivalent.

Since $\mathcal{S}(Z)$ contains an operator equivalent to J_0 for every dense non-compact operator Z, we can use Lemma 3 to show that $\mathcal{S}(A)$ contains the operator $J_0 \oplus J_0$ which is similar to J_0. ∎

LEMMA 5. *For every dense operator A, the semigroup $\mathcal{S}(J_0)$ contains an operator equivalent to A.*

PROOF: We may assume that A is positive. If it is not, replace it by the equivalent operator $\sqrt{AA^*}$. Using spectral theorem we can decompose A as $A = A_1 \oplus A_2$ where A_1 and A_2 are both dense.

In [5, proof of Theorem 3.6] it is shown that every non-closed operator range is a subset of some range of type J_S. Since we can replace A_1 with an equivalent operator, we may assume that $\operatorname{ran} A_1 \subseteq \operatorname{ran} J_0$. By part (i) of Proposition 2 there exists a bounded operator B_1 such that $A_1 = J_0 B_1$.

Let $\{H_n\}$ be the sequence of Proposition 3 subspaces belonging to the range of J_0 and $\{K_n\}$ the corresponding sequence for B_1. For each n we decompose the space $H_n = H_n' \oplus H_n''$ in such a way that $\dim H_n' = \infty$ and $\dim H_n'' = \dim K_n$. According to this decomposition

$$J_0 = \left(\bigoplus_{n=1}^{\infty} 2^{-n} I_{H_n'} \right) \oplus \left(\bigoplus_{n=1}^{\infty} 2^{-n} I_{H_n''} \right)$$

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on the space $\left(\bigoplus_{n=1}^{\infty} H_n'\right) \oplus \left(\bigoplus_{n=1}^{\infty} H_n''\right)$. The first part obviously equals J_0; the second we label C_1. By the remark after Proposition 3, operator C_1 is equivalent to B_1. There exist invertible operators X_1 and Y_1, such that $C_1 = X_1 B_1 Y_1$. Then

$$J_0 = J_0 \oplus C_1 = J_0 \oplus X_1 B_1 Y_1 \sim J_0 \oplus B_1 Y_1 X_1.$$

By analogy we find an operator B_2 such that $A_2 = B_2 J_0$ and show that J_0 is similar to $Y_2 X_2 B_2 \oplus J_0$. Multiplying both operators similar to J_0 we obtain

$$\begin{aligned}(Y_2 X_2 B_2 \oplus J_0)(J_0 \oplus B_1 Y_1 X_1) &= Y_2 X_2 B_2 J_0 \oplus J_0 B_1 Y_1 X_1 \\ &= (Y_2 X_2 \oplus I)(A_2 \oplus A_1)(I \oplus Y_1 X_1).\end{aligned}$$

The last product is obviously equivalent to A. □

Lemma 6. *Every dense operator $A \in B(H)$ can be expressed as a product of six operators $A = T_1 T_2 \dots T_6$ where each T_i is similar to a positive dense operator.*

Proof: Since A is dense, A has the polar decomposition $A = UP$, where U is unitary and P is a positive dense operator. Using [8, Proposition 5], $U = P_1 P_2 \dots P_6$ where P_i are all positive, invertible operators. Now

$$\begin{aligned}UP &= P_1 P_2 \dots P_6 P \\ &= P_1 P_2 \dots P_6 \sqrt{P}\sqrt{P} \\ &\sim \left(P_1^{1/2} P_2 \dots P_6 \sqrt{P} P_1^{-1/2}\right)\left(P_1^{1/2}\sqrt{P} P_1^{1/2}\right)\end{aligned}$$

$T_6 := P_1^{1/2}\sqrt{P}P_1^{1/2}$ is a dense positive operator, while $P_1^{1/2} P_2 \dots P_6 \sqrt{P} P_1^{-1/2}$ is similar to $P_2 P_3 \dots P_6 \sqrt{P}$ which can be expressed as $T_1 \dots T_5$ by repeating the above calculation four more times. □

Theorem 3. *Let A be a non-compact dense operator. The semigroup $\mathcal{S}(A)$ is equal to the set of all dense operators.*

Proof: We have already proved that the set of dense operators is itself a similarity invariant semigroup and therefore contains $\mathcal{S}(A)$.

Now we prove that $\mathcal{S}(A)$ contains every dense operator. We already know that $J_0 \in \mathcal{S}(Z)$ for every dense non-compact operator Z.

Let P be an arbitrary positive, dense operator. It can be decomposed as $P = P_1 \oplus P_2$ where P_1 and P_2 are both dense and positive. By Lemma 5 the semigroup $\mathcal{S}(J_0) = \mathcal{S}(Z)$ contains operators P_1' and P_2' equivalent to P_1 and P_2 respectively. This fulfils the assumptions of Lemma 3. We infer that $P_1 \oplus P_2 \in \mathcal{S}(A)$.

The semigroup $\mathcal{S}(A)$ contains all positive dense operators. Using Lemma 6 we prove that it contains all dense operators. □

Corollary 2. *All proper, similarity invariant subsemigroups of the semigroup of all dense operators are contained in the set of compact operators.*

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4. Operators with finite nullities and co-nullities

In this section we consider those operators with non-closed range that have both nullity and co-nullity nonzero and finite. We shall show that using similarity and multiplication we can force the null-spaces of such operators and their adjoints to behave quite nicely. In this way we shall be able to reduce the problem to the case of a dense operator.

One complication may arise: even though the operator A we started with is not compact, we can destroy the non-compactness by reckless multiplication of similar operators. This would be bad, because the whole theory of the preceding section works only for non-compact operators. Let me first explain how to avoid this problem.

Let A_1 and A_2 be non-compact operators. We know that their product may happen to be compact. On the other hand, we can always replace A_1 and A_2 with similar operators in such a way that their product is non-compact: since A_1 and A_2 are not compact, by [**5**, Theorem 2.5] there exist closed infinite dimensional subspaces $M_1 \subseteq \operatorname{ran} A_1$ and $M_2 \subseteq \operatorname{ran} A_2$. Let $N_1 = A_1^{-1}(M_1)$. Codimension of the space N_1 is infinite because $\dim(\operatorname{ran} A_1/M_1) = \infty$, hence there exists a unitary operator U that maps M_2 onto N_1. The range of $A_1UA_2U^*$ contains M_1 therefore $A_1UA_2U^*$ is non-compact.

Now let A_1 and A_2 be such that A_1A_2 is not compact. If we choose invertible operators T_1 and T_2, when can we be certain that the product

$$B = (T_1A_1T_1^{-1})(T_2A_2T_2^{-1})$$

is still non-compact? One thing is certain: if we make sure that T_1 and T_2 are both in the space $\mathbb{C} + K(H)$, it is obvious that $B = A_1A_2 + K$ where K is a compact operator. Since the operator A_1A_2 is not compact, neither is B.

The null-spaces of operators and their adjoints that we are shuffling around by similarity are all finite dimensional. Therefore we can constrain the similarity coefficients to be equal to identity on some subspace of finite codimension. With this in mind we shall use the above remark implicitly.

Lemma 7. *Let A be a bounded operator with non-closed range on H and* $\operatorname{nul} A$, $\operatorname{nul} A^* < \infty$. *Assume N and R are closed subspaces of H satisfying the conditions*

(i) $\dim N = \operatorname{nul} A$,

(ii) $\dim R^{\perp} = \operatorname{nul} A^*$,

(iii) $\dim(N \cap R) = \dim(\ker A \cap \overline{\operatorname{ran} A})$.

Then there exists an invertible operator $T \in \mathbb{C} + K(H)$ such that

$$\ker(T^{-1}AT) = N \quad \text{and} \quad \overline{\operatorname{ran}(T^{-1}AT)} = R.$$

Proof: Set $N_1 = N \cap R$ and $N_2 = N \ominus N_1$. By analogy, set $N_1' = \ker A \cap \overline{\operatorname{ran} A}$ and $N_2' = \ker A \ominus N_1'$. Moreover let $R_1 = R \ominus N_1$ and $R_1' = \overline{\operatorname{ran} A} \ominus N_1'$. Spaces R_1 and

R_1' have finite codimensions in H hence the space $M = R_1 \cap R_1'$ has finite codimension in both R_1 and R_1'. Let $S = R_1 \ominus M$ and $S' = R_1' \ominus M$. Finally, let $L = (N + R)^\perp$ and $L' = (\ker A + \overline{\operatorname{ran} A})^\perp$.

Space H is the non-orthogonal direct sum $H = L \dotplus N_1 \dotplus N_2 \dotplus S \dotplus M$. On the other hand $H = L' \dotplus N_1' \dotplus N_2' \dotplus S' \dotplus M$. From the assumptions it is easy to verify that every space has equal dimension as its primed counterpart. Moreover, the only infinite dimensional subspace in the above sum is M. Consequently there exists a bounded invertible operator T satisfying equalities $T(L) = L'$, $T(N_i) = N_i'$ for $i = 1, 2$, $T(S) = S'$ and $T|_M = I$. The operator T is in the set $\mathbb{C} + K(H)$ because it is equal to identity on the subspace M of finite codimension. The following identities also hold

$$T(N) = T(N_1) + T(N_2) = \ker A,$$
$$T^{-1}(\overline{\operatorname{ran} A}) = T^{-1}(M) + T^{-1}(S') + T^{-1}(N_1') = R.$$

We see that T is indeed the required operator. ∎

The following is a well known formula for the nullity of the product of operators. It will be used on several occasions throughout this section.

$$\operatorname{nul}(AB) = \operatorname{nul} B + \dim(\operatorname{ran} B \cap \ker A) \tag{1}$$

We shall also need an inequality of this type for the complement of the range

$$\dim(\operatorname{ran} AB)^\perp \leqslant \dim(\operatorname{ran} A)^\perp + \dim(\overline{\operatorname{ran} B} + \ker A)^\perp, \tag{2}$$

which follows from the formula (1) for A^* and B^* considering

$$(\overline{\operatorname{ran} B} + \ker A)^\perp = \ker B^* \cap \overline{\operatorname{ran} A^*}.$$

We obtain inequality instead of equality because $\operatorname{ran} A^* \subsetneqq \overline{\operatorname{ran} A^*}$.

Lemma 8. *Let A_1 and A_2 be non-compact operators with non-closed ranges and finite nullities and co-nullities. If n, $m \in \mathbb{N}$ are such that*

$$\operatorname{nul} A_2 \leqslant n \leqslant \operatorname{nul} A_1 + \operatorname{nul} A_2,$$

$$\operatorname{nul} A_1^* \leqslant m \leqslant \operatorname{nul} A_1^* + \operatorname{nul} A_2^*,$$

there exist operators $B_1 \sim A_1$ and $B_2 \sim A_2$ such that $B_1 B_2$ is not compact, $\operatorname{nul}(B_1B_2) = n$ *and* $\operatorname{nul}(B_1B_2)^* = m$.

Proof: As shown at the beginning of this section, we may assume that $A_1 A_2$ is not compact. Let T be an invertible operator that maps $\ker A_1$ to a subspace of $\overline{\operatorname{ran} A_2}$ and $C = TA_1T^{-1}$. Operator C has the properties

$$\ker C = T \ker A_1 \subseteq \overline{\operatorname{ran} A_2},$$

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$$\overline{\operatorname{ran} C^*} = (\ker C)^\perp \supseteq \overline{\operatorname{ran} A_2}^\perp = \ker A_2.$$

Since $\operatorname{ran} A_2$ is not closed, there exists a subspace $M \subseteq \overline{\operatorname{ran} A_2}$ with dimension $\operatorname{nul} A_1$ such that $\dim(M \cap \operatorname{ran} A_2) = n - \operatorname{nul} A_2$. [This follows from the fact that $\dim(\overline{\mathcal{R}}/\mathcal{R}) = \infty$ for any non-closed operator range $\mathcal{R}$. See [**5**, Corollary of Theorem 2.3].] Choose an invertible operator P that leaves invariant the subspaces $\overline{\operatorname{ran} A_2}$ and $(\operatorname{ran} A_2)^\perp$ such that $P(M) = \ker C$ and let $B_2 = PA_2P^{-1}$. Notice that

$$\operatorname{nul}(CB_2) = \operatorname{nul} A_2 + \dim(\ker C \cap \operatorname{ran} B_2) = \operatorname{nul} A_2 + (n - \operatorname{nul} A_2) = n.$$

Analogously we choose a subspace $N \subseteq \overline{\operatorname{ran} C^*}$ of dimension $\operatorname{nul} A_2^*$ satisfying

$$\dim(N \cap \operatorname{ran} C^*) = m - \operatorname{nul} C^*.$$

Take an invertible operator S that leaves subspaces $\ker C$ and $\overline{\operatorname{ran} C^*}$ invariant and maps N to $\ker B_2^*$. Setting $B_1 = (S^*)^{-1}CS^*$ we infer

$$\operatorname{nul}(B_1B_2)^* = \operatorname{nul} C^* + \dim(N \cap \operatorname{ran} C^*) = m.$$

Note that operators T, P and S can be chosen from the set $\mathbb{C} + K(H)$. In this case, operator B_1B_2 is not compact. $\square$

Lemma 9. *Let A be a non-compact operator with non-closed range, satisfying the inequality $0 < \operatorname{nul} A$, $\operatorname{nul} A^* < \infty$. For any integers $n \geqslant \operatorname{nul} A$ and $m \geqslant \operatorname{nul} A^*$ the semigroup $\mathcal{S}(A)$ contains a non-compact operator B with non-closed range such that $\operatorname{nul} B = n$ and $\operatorname{nul} B^* = m$.*

Proof: Using the previous lemma k times we see that such an operator B exists for integers n, m satisfying $\operatorname{nul} A \leqslant n \leqslant k(\operatorname{nul} A)$ and $\operatorname{nul} A^* \leqslant m \leqslant k(\operatorname{nul} A^*)$. Since $\operatorname{nul} A \neq 0$ and $\operatorname{nul} A^* \neq 0$, n and m can be arbitrarily large provided that we choose a large enough k. $\square$

Lemma 10. *Let A be a non-compact operator with non-closed range and finite nullity and co-nullity. Let X be an operator with non-closed range. Assume that the following conditions hold*

$$\ker A = \ker X,$$

$$\overline{\operatorname{ran} A} = \overline{\operatorname{ran} X},$$

$$\ker A \cap \overline{\operatorname{ran} A} \subseteq \operatorname{ran} A.$$

Then there exists an invertible operator T such that

$$\ker(TAT^{-1}) = \ker X,$$

$$\overline{\operatorname{ran} X} \supseteq \operatorname{ran}(TAT^{-1}) \supseteq \operatorname{ran} X.$$

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PROOF: Let $K = \ker A \cap \overline{\operatorname{ran} A}$ and $L = \overline{\operatorname{ran} A} \ominus K$. Space $L \cap \operatorname{ran} A$ is a dense range of a non-compact operator in L. That it is an operator range follows from the fact that intersection of operator ranges is an operator range and because L has finite codimension in $\operatorname{ran} A$ it is the range of a non-compact operator. To prove the denseness define P_L to be the orthogonal projector on L and notice that since $K \subseteq \operatorname{ran} A$, the space $L \cap \operatorname{ran} A$ equals $P_L(\operatorname{ran} A)$ which is dense in L.

Space $R = (K + \operatorname{ran} X) \cap L$ is another operator range in L. By [5, proof of Theorem 3.6] there exists a subrange of $L \cap \operatorname{ran} A$ which is unitarily equivalent to R. Hence there exists unitary operator U on L that maps $L \cap \operatorname{ran} A$ to a superset of R. By setting $U|_K = I$ we extend U to a unitary operator on $\overline{\operatorname{ran} A}$. Using the modularity law along with the facts $K \subseteq \operatorname{ran} A$, $K + L = \overline{\operatorname{ran} A}$ we obtain $K + (L \cap \operatorname{ran} A) = \operatorname{ran} A$ and $K + R = K + \operatorname{ran} X$ therefore

$$U(\operatorname{ran} A) = U\big(K + (L \cap \operatorname{ran} A)\big) \supseteq K + R \supseteq \operatorname{ran} X.$$

To complete the proof we extend U to an operator T on H in such a way that $T(\ker X) = \ker X$. ∎

LEMMA 11. *Let X be an operator with non-closed range that has finite nullity and co-nullity. If A_1 and A_2 are non-compact operators with non-closed ranges satisfying $\operatorname{nul} A_1 = \operatorname{nul} A_2 = \operatorname{nul} X$ and $\operatorname{nul} A_1^* = \operatorname{nul} A_2^* = \operatorname{nul} X^*$, there exist operators $B_1 \sim A_1$ and $B_2 \sim A_2$ such that*

$$\ker(B_1 B_2) = \ker X,$$

$$\overline{\operatorname{ran} X} \supseteq \operatorname{ran} B_1 B_2 \supseteq \operatorname{ran} X.$$

PROOF: We may assume that $A_1 A_2$ is not compact. Let $J_1 = \overline{\operatorname{ran} X}$ and $K_2 = \ker X$. It is easy to verify that there exists a closed subspace $J_2 \leqslant H$ such that $J_1 + J_2 = H$, $\dim J_2^\perp = \operatorname{nul} A_2^*$ and

$$\dim(J_2 \cap K_2) = \dim(\ker A_2 \cap \overline{\operatorname{ran} A_2}).$$

Using Lemma 7 we obtain an operator C_2 similar to A_2 such that $\ker C_2 = K_2$ and $\overline{\operatorname{ran} C_2} = J_2$.

Assume for a moment that there exists a space $K_1 \leqslant H$ with dimension $\operatorname{nul} A_1$ satisfying the requirements

(i) $\dim(J_1 \cap K_1) = \dim(\ker A_1 \cap \overline{\operatorname{ran} A_1}) =: n$,

(ii) $K_1 \cap \operatorname{ran} C_2 = 0$,

(iii) $K_1 + J_2 = H$,

(iv) $K_1 \cap K_2 = 0$.

By (i) and Lemma 7 there exists an operator $C_1 \sim A_1$ that has K_1 as the null-space and J_1 as the closure of the range. We use (ii) together with the formula for the nullity of

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the product

$$\operatorname{nul}(C_1C_2) = \operatorname{nul} C_2 + \dim(\operatorname{ran} C_2 \cap \ker C_1)$$

to show that $\operatorname{nul}(C_1C_2) = \operatorname{nul} C_2$ therefore $\ker(C_1C_2) = \ker C_2 = K_2$. Now use the inequality (2) for the complements of range

$$\dim(\operatorname{ran} C_1C_2)^{\perp} \leqslant \dim(\operatorname{ran} C_1)^{\perp} + \dim(\overline{\operatorname{ran} C_2} + \ker C_1)^{\perp}.$$

From here and (iii) we infer that $\overline{\operatorname{ran}(C_1C_2)} = \overline{\operatorname{ran} C_1} = J_1$. The statement (iv) will be needed later in the proof.

Now we prove that the space K_1 exists. Because we want $K_1 + J_2 = H$ the equation

$$\dim(K_1 \cap J_2) = \operatorname{nul} A_1 - \dim J_2^{\perp} =: m$$

will have to hold. But $K_1 + J_1 \subseteq H$ and since $\dim J_1^{\perp} = \dim J_2^{\perp}$ it will also be true that

$$\dim(K_1 \cap J_1) = n \geqslant \dim(K_1 \cap J_2).$$

Space $J_1 \cap J_2 \cap K_2^{\perp}$ has finite codimension therefore we can choose m-dimensional subspace K_1' of $J_1 \cap J_2 \cap K_2^{\perp}$ such that $K_1' \cap \operatorname{ran} C_2 = 0$. Then we can extend K_1' to a n-dimensional space $K_1'' \subseteq J_1$ in such a way that $K_1'' \cap K_2 = 0$ and $K_1'' \cap J_2 = K_1'$. Finally, extend K_1'' to a $(\operatorname{nul} A_1)$-dimensional subspace K_1 in H satisfying $K_1 \cap J_1 = K_1''$ and $K_1 \cap K_2 = 0$. From the fact that $K_1 \cap J_2 = K_1'$ we infer $K_1 \cap \operatorname{ran} C_2 = 0$ and $K_1 + J_2 = H$ (the last equality is proved by calculating the codimension). The space K_1 constructed in this way satisfies the conditions (i)–(iv).

There exist subspaces $N_1, N_2 \subseteq J_1 \cap J_2$ with dimension $\dim(J_1 \cap K_2)$ such that $N_1 \cap N_2 = 0$, $C_1(N_1) = N_2$ and

$$(N_1 + N_2) \cap (K_1 + K_2) = 0.$$

How do we know that? Since $C_1 \notin \mathbb{C} + K(H)$, by [1, Corollary 3.4] there exists a subspace M with infinite dimension such that $C_1(M) \cap M = 0$. Because $J_1 \cap J_2$ has finite codimension,

$$M' = M \cap (J_1 \cap J_2) \cap (K_1 + K_2)^{\perp}$$

is still infinite dimensional. The space

$$M'' = M' \cap C_1^{-1}\big(C_1(M') \cap (J_1 \cap J_2) \cap (K_1 + K_2)^{\perp}\big)$$

is also infinite dimensional. Now any subspace $N_1 \subseteq M''$ with the same dimension as $J_1 \cap K_2$ and corresponding $N_2 = C_1(N_1)$ will do.

By a simple calculation (similar to the proof of Lemma 7), there exists an invertible operator $T \in \mathbb{C} + K(H)$ such that

$$T(N_1) = N_1,\ T(N_2) = J_1 \cap K_2,\ T(J_1) = J_1$$

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and

$$T(K_1) = K_1.$$

Define $B_1 = TC_1T^{-1}$. As for C_1 we can prove that $\ker B_1 = K_1$ and $\overline{\operatorname{ran} B_1} = J_1$. The following equality also holds

$$B_1(N_1) = TC_1T^{-1}(N_1) = TC_1(N_1) = J_1 \cap K_2.$$

Moreover, there exists an invertible operator $S \in \mathbb{C} + K(H)$ such that

$$S(K_1) = K_1,\ S(K_2) = K_2,\ S(J_2) = J_2$$

and

$$S(N_1) \subseteq \operatorname{ran} C_2.$$

Define $B_2 = S^{-1}C_2S$. Then

$$\ker B_2 = K_2,\ \overline{\operatorname{ran} B_2} = J_2,\ \operatorname{ran} B_2 \cap K_1 = 0$$

and

$$\operatorname{ran} B_2 \supseteq N_1.$$

As for C_1C_2 we see that $\ker(B_1B_2) = K_2$ and $\overline{\operatorname{ran}(B_1B_2)} = J_1$. What we gained is that

$$\operatorname{ran}(B_1B_2) \supseteq B_1(N_1) = J_1 \cap K_2 = \ker(B_1B_2) \cap \overline{\operatorname{ran}(B_1B_2)}.$$

Operator B_1B_2 clearly fulfills all the requirements of Lemma 10 which completes this proof. ∎

Theorem 4. *Let A be a non-compact operator with non-closed range such that $0 < \operatorname{nul} A, \operatorname{nul} A^* < \infty$. The semigroup $\mathcal{S}(A)$ is equal to the set $\mathcal{M}$ of all bounded operators X with non-closed range that satisfy inequalities*

$$\operatorname{nul} A \leqslant \operatorname{nul} X < \infty \quad \text{and} \quad \operatorname{nul} A^* \leqslant \operatorname{nul} X^* < \infty.$$

Proof: First we prove that $\mathcal{M}$ is a similarity invariant semigroup. It is obvious that any finite product X of operators similar to A satisfies inequality $\operatorname{nul} A \leqslant \operatorname{nul} X < \infty$ and the analogous inequality for adjoints. Moreover, every such product will have non-closed range; if its range is closed it is a Fredholm operator and cannot be expressed as a product of non-Fredholm operators. Hence the semigroup $\mathcal{S}(A)$ is contained in $\mathcal{M}$.

To prove $\mathcal{S}(A) \supseteq \mathcal{M}$ we first show that the semigroup $\mathcal{S}(A)$ contains every operator $Z \in \mathcal{M}$ with $\ker Z = \ker Z^*$. Choose an integer $n \geqslant \max\{\operatorname{nul} A, \operatorname{nul} A^*\}$. By Lemma 9, $\mathcal{S}(A)$ contains a non-compact operator Y' with $\operatorname{nul} Y' = \operatorname{nul} Y'^* = n$. Now it follows from Lemma 11 that $\mathcal{S}(A)$ contains a non-compact operator Y with the property

$$\ker Y = \ker Z = \ker Z^* = \ker Y^*.$$

Every such operator Y is similar to a block diagonal operator $Y_1 \oplus 0 \in B(H \oplus \ker Y)$ where Y_1 is an injective operator with dense range. Using the results of the previous section, $\mathcal{S}(A)$ contains all operators of the form $Z' \oplus 0 \in B(H \oplus \ker Y)$. Operator Z is similar to an operator of this form.

Let X be any operator in $\mathcal{M}$. We may assume that $\operatorname{nul} X \geqslant \operatorname{nul} X^*$; otherwise we prove that $X^* \in \mathcal{S}(A^*)$ which is enough, since it is easy to check that $\mathcal{S}(A^*) = \mathcal{S}(A)^*$.

There exists an operator X' with the same null-space and closure of range as X, while its range is non-closed and strictly contains the range of X. For example, take any $v \in (\overline{\operatorname{ran} X} \setminus \operatorname{ran} X)$ and choose an appropriate operator with the range $\operatorname{ran} X + \operatorname{Lin}\{v\}$.

Using Lemma 9 we obtain a non-compact operator $B \in \mathcal{S}(A)$ that satisfies the conditions $\operatorname{nul} B = \operatorname{nul} X$ and $\operatorname{nul} B^* = \operatorname{nul} X^*$. Now use Lemma 11 on two operators similar to B to see that there exists an operator $C \in \mathcal{S}(A)$ with the properties $\ker C = \ker X$ and $\operatorname{ran} C \supseteq \operatorname{ran} X'$.

By Proposition 2 there exists a unique operator Z satisfying $\ker Z = \ker X$ and $\ker Z^* = \ker X$ such that $X = CZ$. The range of Z is non-closed because $\operatorname{ran} C$ properly contains $\operatorname{ran} X$, while $\ker C \cap \overline{\operatorname{ran} Z} = 0$. As shown in the first part of this proof, every such operator Z is an element of $\mathcal{S}(A)$. This proves that $X \in \mathcal{S}(A)$ and finally $\mathcal{M} \subseteq \mathcal{S}(A)$. ∎

5. Conclusion

Proof: [Proof of Theorem 1] Since $\operatorname{nul} A \approx \operatorname{nul} A^*$ there are three possibilities. When $\operatorname{nul} A = \operatorname{nul} A^* = 0$, operator A is a dense operator and we use Theorem 3. The case $\operatorname{nul} A = \operatorname{nul} A^* = \infty$ is handled by Theorem 2. Theorem 4 solves the problem in the remaining case when $0 < \operatorname{nul} A$, $\operatorname{nul} A^* < \infty$. It is trivial to see that the results of all three theorems agree with Theorem 1. ∎

As mentioned in the introduction, the case of the operator A with non-closed range and $\operatorname{nul} A \not\approx \operatorname{nul} A^*$ remains open. More verbosely, such operators satisfy either

$$\operatorname{nul} A = 0 \quad \text{and} \quad \operatorname{nul} A^* \neq 0 \tag{3}$$

or

$$\operatorname{nul} A < \infty \quad \text{and} \quad \operatorname{nul} A^* = \infty \tag{4}$$

or any of the dual conditons. In fact the second case can be reduced to the first, therefore it is enough to consider operators satisfying the condition (3).

We cannot hope to reduce an operator of this kind to a direct sum of 0 and a dense operator as we did in the preceding section. On the other hand, most of the statements used in Section 3 for dense operators can be adapted to any injective non-Fredholm operators. The only real problem is that there is no obvious analogue of Lemma 6.

Of course, Lemma 6 is too strong for our purposes: all that is needed is a proof that every operator with non-closed range for which (3) holds is expressible as a product of operators of the form $A_1 \oplus A_2$, where A_1 and A_2 are also operators with non-closed range that satisfy the condition (3).

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