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MEAN RESIDUAL LIFETIMES OF CONSECUTIVE-k-OUT-OF-n SYSTEMS

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Abstract

In this paper we study reliability properties of consecutive-k-out-of-n systems with exchangeable components. For $2k \geq n$, we show that the reliability functions of these systems can be written as negative mixtures (i.e. mixtures with some negative weights) of two series (or parallel) systems. Some monotonicity and asymptotic properties for the mean residual lifetime function are obtained and some ordering properties between these systems are established. We prove that, under some assumptions, the mean residual lifetime function of the consecutive-k-out-of-$n : G$ system (i.e. a system that functions if and only if at least k consecutive components function) is asymptotically equivalent to that of a series system with k components. When the components are independent and identically distributed, we show that consecutive-k-out-of-n systems are ordered in the likelihood ratio order and, hence, in the mean residual lifetime order, for $2k \geq n$. However, we show that this is not necessarily true when the components are dependent.

Keywords: Consecutive-k-out-of-n system; exchangeable distribution; signature; mean residual lifetime; stochastic order

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1. Introduction and preliminaries

A linear consecutive-k-out-of-$n : F$ system consists of n linearly ordered components such that the system fails if and only if at least k consecutive components fail. A linear consecutive-k-out-of-$n : G$ system, on the other hand, consists of n linearly ordered components such that the system functions if and only if at least k consecutive components function. We denote the consecutive-k-out-of-$n : F$ and consecutive-k-out-of-$n : G$ systems by $(C, k, n : F)$ and $(C, k, n : G)$, respectively.

Obviously, the series system is represented by $(C, n, n : G) = (C, 1, n : F)$ and the parallel system by $(C, n, n : F) = (C, 1, n : G)$. A $(C, k, n : F)$ system is the dual of a $(C, k, n : G)$ system. The definition of dual systems can be found in Barlow and Proschan (1975, p. 12). Let X_i denote the state of component i ($X_i = 0$ if component i has failed and $X_i = 1$ if component i is working). Then the structure function of the $(C, k, n : F)$ system is given by

$$\phi_{k\,|\,n:F}(X_1, X_2, \dots, X_n) = \prod_{j=1}^{n-k+1} \left\{1 - \prod_{i=j}^{j+k-1} (1 - X_i)\right\}.$$

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Let $\phi_{k\,|\,n:G}$ denote the structure function of the $(C, k, n : G)$ system. Then, using the relation between the structure functions of a system and its dual system, we have

$$\phi_{k\,|\,n:G}(X_1, X_2, \dots, X_n) = 1 - \phi_{k\,|\,n:F}(1-X_1, 1-X_2, \dots, 1-X_n) = 1 - \prod_{j=1}^{n-k+1}\left\{1 - \prod_{i=j}^{j+k-1} X_i\right\}.$$

It is well known that the reliabilities of consecutive-k-out-of-n systems are characterized by the longest-run random variable based on a binary sequence of random variables $X_1, X_2, \dots, X_n$. More explicitly, the reliabilities of $(C, k, n : F)$ and $(C, k, n : G)$ systems are given by

$$R_{n\,|\,k:F} = \mathrm{P}\{L_n^0 < k\} \quad \text{and} \quad R_{n\,|\,k:G} = \mathrm{P}\{L_n^1 \ge k\},$$

respectively, where L_n^0 and L_n^1 represent the longest run of failures and the longest run of successes, respectively, in the sequence $X_1, X_2, \dots, X_n$. Exact expressions, bounds, and approximations for $R_{n\,|\,k:F}$ and $R_{n\,|\,k:G}$ have been well studied in the literature. We refer the reader to Boland and Papastavridis (1999), Balakrishnan and Koutras (2002), Fu and Lou (2003), Boland and Samaniego (2004), and Eryilmaz (2005).

The lifetime distribution of the $(C, k, n : F)$ system has been discussed in papers such as Derman *et al.* (1982), Shanthikumar (1985), Chen and Hwang (1985), and Aki and Hirano (1996). Let T_i be the lifetime of the ith component and assume that $T_1, T_2, \dots, T_n$ are independent and identically distributed (i.i.d.) random variables with cumulative distribution function (CDF) $F(t) = \mathrm{P}\{T_i \le t\}$. As proved in Aki and Hirano (1996), the CDF of the lifetime of a $(C, k, n : F)$ system is written as a finite mixture of the distributions of order statistics of the lifetimes of components. Let $T_{k\,|\,n:F}$ denote the lifetime of the $(C, k, n : F)$ system. Then its CDF can be written as

$$F_{k\,|\,n:F}(t) = \mathrm{P}\{T_{k\,|\,n:F} \le t\} = \sum_{i=1}^{n} \omega_i F_{i:n}(t),$$

where $F_{i:n}(t)$ is the CDF of the ith order statistic, $T_{i:n}$, in the random sample $T_1, T_2, \dots, T_n$ (or the lifetime of an $(n-i+1)$-out-of-n system with component lifetimes $T_1, T_2, \dots, T_n$) and

$$\omega_i = \frac{r_{i,k}}{n(n-1)\cdots(n-i+1)} \tag{1.1}$$

with

$$r_{m,k} = (m-1)!\,(n-m+1)N_k(n, m-1) - m!\,N_k(n, m)$$

and

$$N_k(n, m) = \sum_{i=0}^{\lfloor n/k \rfloor} (-1)^i \binom{n-m+1}{i}\binom{n-ki}{n-m},$$

where $\lfloor x \rfloor$ denotes the integer part of x.

Samaniego (1985) (see also Kochar *et al.* (1999)) introduced the concept of the 'signature' of a coherent system. He proved that any coherent system with lifetime T and component lifetimes $T_1, T_2, \dots, T_n$ having absolutely continuous distributions satisfies

$$\mathrm{P}\{T > t\} = \sum_{i=1}^{n} p_i\, \mathrm{P}\{T_{i:n} > t\},$$

where p_i is the probability that the system fails upon the occurrence of the ith component failure, i.e. $p_i = \mathrm{P}\{T = T_{i:n}\}$. The vector $\boldsymbol{p} = (p_1, p_2, \dots, p_n)$ is called the system's signature. The ith element of this vector can be expressed as

$$p_i = \frac{\text{number of orderings for which the } i\text{th failure causes system failure}}{n!},$$

for $i = 1, 2, \dots, n$.

It was also proved in Kochar *et al.* (1999) that if $\boldsymbol{p}$ denotes the signature of a system with structure function ϕ and whose components have i.i.d. lifetimes, and if $\boldsymbol{p}^{\mathrm{D}}$ is the signature of the dual system, ϕ^{D}, then $p_i = p^{\mathrm{D}}_{n-i+1}$ for $i = 1, 2, \dots, n$.

Navarro *et al.* (2005) (see also Navarro and Rychlik (2006)), proved that this representation holds for systems with possibly dependent components when the random vector of component lifetimes, $(T_1, T_2, \dots, T_n)$, has an absolutely continuous, exchangeable distribution (i.e. its joint distribution is invariant under permutation of the variables).

The following example illustrates the computation of $\boldsymbol{p}$ for a consecutive-k-out-of-n system.

Example 1.1. Consider the $(C, 2, 3 : F)$ system whose lifetime is given by

$$T_{2\,|\,3:F} = \min(\max(T_1, T_2), \max(T_2, T_3)).$$

The order statistic representation of $T_{2\,|\,3:F}$ for each possible ordering is as follows:

$$\begin{aligned}
T_1 < T_2 < T_3 \quad &\Longrightarrow \quad T_{2\,|\,3:F} = T_{2:3},\\
T_1 < T_3 < T_2 \quad &\Longrightarrow \quad T_{2\,|\,3:F} = T_{3:3},\\
T_2 < T_1 < T_3 \quad &\Longrightarrow \quad T_{2\,|\,3:F} = T_{2:3},\\
T_2 < T_3 < T_1 \quad &\Longrightarrow \quad T_{2\,|\,3:F} = T_{2:3},\\
T_3 < T_1 < T_2 \quad &\Longrightarrow \quad T_{2\,|\,3:F} = T_{3:3},\\
T_3 < T_2 < T_1 \quad &\Longrightarrow \quad T_{2\,|\,3:F} = T_{2:3}.
\end{aligned}$$

If (T_1, T_2, T_3) has an absolutely continuous, exchangeable distribution, then

$$\mathrm{P}\{T_1 < T_2 < T_3\} = \mathrm{P}\{T_{\sigma(1)} < T_{\sigma(2)} < T_{\sigma(3)}\} = \tfrac{1}{6}$$

for any permutation σ. Therefore, the signature of the $(C, 2, 3 : F)$ system is $\boldsymbol{p} = (0, \frac{4}{6}, \frac{2}{6})$.

Since the $(C, 2, 3 : G)$ system is the dual of the $(C, 2, 3 : F)$ system, the signature of the $(C, 2, 3 : G)$ system is $\boldsymbol{p} = (\frac{2}{6}, \frac{4}{6}, 0)$. Table 1 displays the signatures of the $(C, k, n : F)$ and $(C, k, n : G)$ systems for various values of k and n.

It is not difficult to see that the set of coefficients given by (1.1) is actually the signature of the $(C, k, n : F)$ system. This implies that the lifetime distribution of the $(C, k, n : G)$ system can be written as

$$F_{k\,|\,n:G}(t) = \mathrm{P}\{T_{k\,|\,n:G} \le t\} = \sum_{i=1}^{n} \omega_{n-i+1} F_{i:n}(t) \tag{1.2}$$

for $k = 1, 2, \dots, n$. Moreover, from Navarro *et al.* (2005) and Navarro and Rychlik (2006), both representations hold for consecutive-k-out-of-n systems if their component lifetimes have an absolutely continuous, exchangeable joint distribution.

TABLE 1: Signatures of consecutive-k-out-of-n systems.

System	$\boldsymbol{p}$
$(C, 2, 4 : F)$	$(0, \frac{1}{2}, \frac{1}{2}, 0)$
$(C, 2, 4 : G)$	$(0, \frac{1}{2}, \frac{1}{2}, 0)$
$(C, 2, 5 : F)$	$(0, \frac{4}{10}, \frac{5}{10}, \frac{1}{10}, 0)$
$(C, 2, 5 : G)$	$(0, \frac{1}{10}, \frac{5}{10}, \frac{4}{10}, 0)$
$(C, 3, 5 : F)$	$(0, 0, \frac{3}{10}, \frac{5}{10}, \frac{2}{10})$
$(C, 3, 5 : G)$	$(\frac{2}{10}, \frac{5}{10}, \frac{3}{10}, 0, 0)$
$(C, 2, 6 : F)$	$(0, \frac{5}{15}, \frac{7}{15}, \frac{3}{15}, 0, 0)$
$(C, 2, 6 : G)$	$(0, 0, \frac{3}{15}, \frac{7}{15}, \frac{5}{15}, 0)$
$(C, 3, 6 : F)$	$(0, 0, \frac{2}{10}, \frac{4}{10}, \frac{4}{10}, 0)$
$(C, 3, 6 : G)$	$(0, \frac{4}{10}, \frac{4}{10}, \frac{2}{10}, 0, 0)$

2. Mean residual lifetime function

Let T denote the lifetime of the system. Then the residual lifetime of the system given that the system has survived up to time t is $[T - t \mid T > t]$. The mean residual lifetime (MRL) function, defined by $m_T(t) = \mathrm{E}(T - t \mid T > t)$, plays an important role in reliability and survival analysis. It can be computed from

$$m_T(t) = \frac{1}{R_T(t)} \int_t^\infty R_T(x)\, \mathrm{d}x, \tag{2.1}$$

where $R_T(t) = \mathrm{P}\{T > t\}$ is the reliability (or survival) function of T.

The MRL function of the $(C, k, n : F)$ system is then defined by

$$m_{k \mid n:F}(t) = \mathrm{E}(T_{k \mid n:F} - t \mid T_{k \mid n:F} > t).$$

The MRL function of the $(C, k, n : G)$ system is defined in a similar way.

From (1.1) and (2.1), the MRL function of the $(C, k, n : F)$ system is given by

$$m_{k \mid n:F}(t) = \frac{\sum_{i=1}^n \omega_i R_{i:n}(t) m_{i:n}(t)}{\sum_{i=1}^n \omega_i R_{i:n}(t)},$$

where $R_{i:n}$ denotes the reliability function of the ith order statistic and

$$m_{i:n}(t) = \mathrm{E}(T_{i:n} - t \mid T_{i:n} > t)$$

denotes the MRL function of the ith order statistic (or the $(n - i + 1)$-out-of-n system). Note that this expression also holds for systems with possibly dependent components if they have an exchangeable and absolutely continuous joint distribution. A similar expression can be obtained for the MRL function of the $(C, k, n : G)$ system by using (1.2).

To obtain a simpler formula for the MRL function of the $(C, k, n : F)$ system for $2k \geq n$, we need the following lemma.

Lemma 2.1. *For $2k \geq n$, the CDF of the $(C, k, n : F)$ system having i.i.d. components with CDF $F(t)$ is given by*

$$F_{k\,|\,n:F}(t) = (n-k+1)F_{k:k}(t) - (n-k)F_{k+1:k+1}(t), \tag{2.2}$$

where $F_{k:k}(t)$ denotes the CDF of the parallel system having k i.i.d. components with CDF $F(t)$.

Proof. The reliability of the $(C, k, n : F)$ system for $2k \geq n$ is given by

$$R_{n\,|\,k:F} = \mathrm{P}\{L_n^0 < k\} = 1 - (n-k+1)q^k + (n-k)q^{k+1},$$

where $q = \mathrm{P}\{X_i = 0\}$ (see Tong (1985)). Using the relation

$$\mathrm{P}\{T_{k\,|\,n:F} > t\} = \mathrm{P}\{L_n^0(t) < k\}$$

with $q = \mathrm{P}\{T_i \leq t\} = F(t)$, the CDF of $T_{k\,|\,n:F}$ for $2k \geq n$ can be written as

$$F_{k\,|\,n:F}(t) = (n-k+1)F^k(t) - (n-k)F^{k+1}(t),$$

and the proof is complete.

Proposition 2.1. *For $2k \geq n$, the MRL function of the $(C, k, n : F)$ system having i.i.d. components with CDF $F(t)$ is given by*

$$m_{k\,|\,n:F}(t) = \frac{(n-k+1)(1-F^k(t))m_{k:k}(t) - (n-k)(1-F^{k+1}(t))m_{k+1:k+1}(t)}{1-(n-k+1)F^k(t)+(n-k)F^{k+1}(t)},$$

where $m_{k:k}(t)$ denotes the MRL function of the parallel system having k i.i.d. components with CDF $F(t)$.

The proof is immediate from (2.1) and the preceding lemma, using the fact that $F_{k:k} = F^k$.

Analogously, for $2k \geq n$ the CDF of the $(C, k, n : G)$ system with i.i.d. components with CDF $F(t)$ is given by

$$F_{k\,|\,n:G}(t) = (n-k+1)F_{1:k}(t) - (n-k)F_{1:k+1}(t) \tag{2.3}$$

and its MRL function is given by

$$m_{k\,|\,n:G}(t) = \frac{(n-k+1)R^k(t)m_{1:k}(t) - (n-k)R^{k+1}(t)m_{1:k+1}(t)}{(n-k+1)R^k(t)-(n-k)R^{k+1}(t)},$$

where $R(t) = 1 - F(t)$ and $F_{1:k}(t)$ and $m_{1:k}(t)$ respectively denote the CDF and the MRL function of a series system having k i.i.d. components with CDF $F(t)$.

Figures 1 and 2 respectively show the graphs of the MRL functions of the $(C, 3, 5 : F)$ and $(C, 4, 5 : F)$ systems under the classical assumption that the components are exponential with CDF $F(t) = 1 - \exp(-\lambda t)$, for $t > 0$ and $\lambda = 2, 3$. Note that the MRL functions given in the figures are ordered in λ and that the MRL functions given in Figure 2 are decreasing. However, the MRL functions given in Figure 1 are bathtub shaped (i.e. they first decrease and then increase as t increases). Also note that they are equivalent to $1/\lambda$ (i.e. the MRL function of an exponential component) in the limit as $t \to \infty$. We shall see that this is a general property.

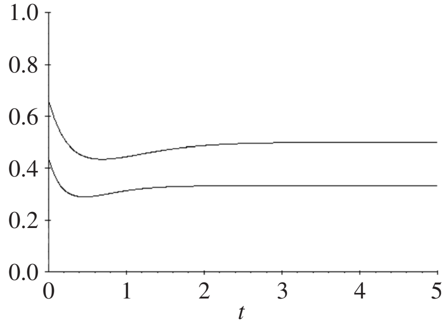

FIGURE 1: MRL functions of $(C, 3, 5 : F)$ systems having exponential components with $\lambda = 2$ (*upper line*) and $\lambda = 3$ (*lower line*).

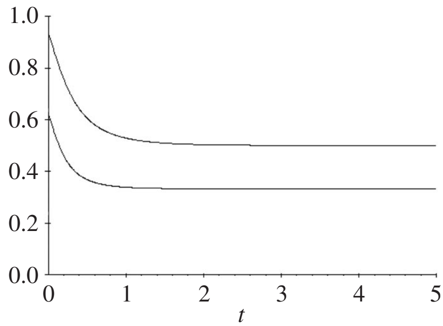

FIGURE 2: MRL functions of $(C, 4, 5 : F)$ systems having exponential components with $\lambda = 2$ (*upper line*) and $\lambda = 3$ (*lower line*).

3. Ordering properties

The main stochastic orderings between random variables are the (usual) stochastic order (st), the likelihood ratio order (lr), the hazard rate order (hr) and the mean residual lifetime order (mrl), respectively defined by

$$\begin{aligned}
X \leq_{\text{st}} Y &\iff R_X(t) \leq R_Y(t) \text{ for all } t,\\
X \leq_{\text{lr}} Y &\iff f_X(t)/f_Y(t) \text{ is decreasing for all } t,\\
X \leq_{\text{hr}} Y &\iff h_X(t) \geq h_Y(t) \text{ for all } t,\\
X \leq_{\text{mrl}} Y &\iff m_X(t) \leq m_Y(t) \text{ for all } t,
\end{aligned}$$

where $R(t)$ is the reliability function, $f(t)$ is the density function, $h(t)$ is the hazard rate function ($h(t) = f(t)/R(t)$), and $m(t)$ is the MRL function. Throughout the paper, 'increasing' and 'decreasing' will be used in the weak sense and $a/0 = \infty$ for $a > 0$. Properties and applications of these orderings can be found in Shaked and Shanthikumar (1994). In particular, they satisfy the following general relations (the complete diagram can be seen in Navarro *et al.* (1997)):

$$\begin{array}{ccccc}
 & & X \leq_{\text{mrl}} Y & \Longrightarrow & \mathrm{E}(X) \leq \mathrm{E}(Y)\\
 & & \Uparrow & & \Uparrow\\
X \leq_{\text{lr}} Y & \Longrightarrow & X \leq_{\text{hr}} Y & \Longrightarrow & X \leq_{\text{st}} Y
\end{array}$$

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To obtain ordering properties between consecutive-k-out-of-n systems, we shall use the concepts of *path sets* and *cut sets* of a coherent system. Let ϕ be the structure function of a coherent system with n components. A path set is a set $P \subseteq \{1, 2, \dots, n\}$ such that $\phi(x_1, x_2, \dots, x_n) = 1$ if $x_i = 1$ for $i \in P$, and a cut set is a set $P \subseteq \{1, 2, \dots, n\}$ such that $\phi(x_1, x_2, \dots, x_n) = 0$ if $x_i = 1$ for $i \in P$ (see Barlow and Proschan (1975, p. 9)). A path set or cut set is said to be minimal if it does not contain proper path sets or cut sets, respectively. Physically, a minimal path set or cut set is a minimal set of elements whose respective functioning or failure ensures the respective functioning or failure of the system.

Obviously, the minimal path sets of a consecutive-k-out-of-$n : G$ system, and the minimal cut sets of a consecutive-k-out-of-$n : F$ system, are $P_1 = \{1, 2, \dots, k\}$, $P_2 = \{2, 3, \dots, k+1\}$, $\dots$, $P_{n-k+1} = \{n-k+1, n-k+2, \dots, n\}$. Analogously, the minimal path sets of a k-out-of-n system are all the sets with k elements. Hence, as $T_{k\,|\,n:G} \geq T_{k+1\,|\,n:G}$ almost surely, $T_{k\,|\,n:G} \geq_{\text{st}} T_{k+1\,|\,n:G}$ according to Theorem 1.A.1 of Shaked and Shanthikumar (1994, p. 5). In a similar way, we obtain $T_{k\,|\,n:F} \leq_{\text{st}} T_{k+1\,|\,n:F}$. Boland and Samaniego (2004) showed that if the components are independent, then $T_{k\,|\,n:F} \geq_{\text{st}} T_{k\,|\,n+1:F}$. Therefore, $T_{k\,|\,n:G} \leq_{\text{st}} T_{k\,|\,n+1:G}$.

The lifetime T of a coherent system can be represented (see Barlow and Proschan (1975, p. 12)) in terms of its minimal path sets, $P_1, P_2, \dots, P_s$, as

$$T = \max_{1 \leq j \leq s} \min_{i \in P_j} T_i,$$

where T_i represents the lifetime of the ith component. Analogously, it can be represented in terms of its minimal cut sets, $C_1, C_2, \dots, C_r$, as

$$T = \min_{1 \leq j \leq r} \max_{i \in C_j} T_i.$$

Therefore, by using the inclusion–exclusion formula (see, e.g. Barlow and Proschan (1975, pp. 25–26)), the reliability function of the system can be expressed as

$$R_T(t) = \mathrm{P}\Bigg\{\bigcup_{1 \leq j \leq s} \{\min(T_i, i \in P_j) > t\}\Bigg\} = \sum_A (-1)^{1+|A|}\, \mathrm{P}\Bigg\{\min\Bigg(T_i, i \in \bigcup_{j \in A} P_j\Bigg) > t\Bigg\} \tag{3.1}$$

(see Block *et al.* (2003)), where $P_1, P_2, \dots, P_s$ are the minimal path sets of T, $A \subseteq \{1, 2, \dots, s\}$, and $|A|$ is the number of elements in A. Note that the function

$$R_{P_A}(t) = \mathrm{P}\Bigg\{\min\Bigg(T_i, i \in \bigcup_{j \in A} P_j\Bigg) > t\Bigg\}$$

is the reliability function of the series system with lifetime $T_{P_A} = \min(T_i, i \in \bigcup_{j \in A} P_j)$, obtained with the components in path set $P_A = \bigcup_{j \in A} P_j$. A similar representation can be obtained in terms of the minimal cut sets and the reliability functions of parallel systems (see Navarro *et al.* (2007)). In particular, $R_{k\,|\,n:G}$, the reliability function of the $(C, k, n : G)$ system, can be written as

$$R_{k\,|\,n:G}(t) = \sum_{i=1}^{n-k+1} R_{P_i}(t) - \sum_{1 \leq i < j \leq n-k+1} R_{P_i \cup P_j}(t) + \cdots + (-1)^{n-k} R_{P_1 \cup P_2 \cup \cdots \cup P_{n-k+1}}(t) \tag{3.2}$$

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for $k = 1, 2, \dots, n$, where $P_i = \{i, i+1, \dots, k+i-1\}$, $i = 1, 2, \dots, n-k+1$, are the minimal path sets of the $(C, k, n : G)$ system and $R_P(t) = \mathrm{P}\{T_j > t : j \in P\}$ is the reliability function of the series system with components in the set P. A similar representation can be obtained for the $(C, k, n : F)$ systems by using the parallel systems.

In particular, if we assume that the lifetimes of the components $(T_1, T_2, \dots, T_n)$ have an exchangeable joint distribution, then the reliability function of a series or parallel system only depends on the number of elements in that system; hence, the reliability function of T can be expressed as

$$R_T(t) = \sum_{i=1}^{n} a_i R_{1:i}(t) = \sum_{i=1}^{n} b_i R_{i:i}(t),$$

for some (unique) real numbers a_i and b_i which do not depend on the joint distribution of $(T_1, T_2, \dots, T_n)$ and satisfy $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 1$. In Navarro *et al.* (2007) (see also Navarro and Shaked (2006)), the vectors $\boldsymbol{a} = (a_1, a_2, \dots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \dots, b_n)$ are respectively called the *minimal* and *maximal signatures* of the system.

If the components are exchangeable then, for the $(C, k, n : G)$ systems, from (3.2) we have

$$R_{k\,|\,n:G}(t) = (n-k+1)R_{1:k}(t) + a_{k+1}R_{1:k+1}(t) + \cdots + a_n R_{1:n}(t) \tag{3.3}$$

for $k = 1, 2, \dots, n$. Hence, its *minimal signature* takes the form $\boldsymbol{a} = (0, \dots, 0, a_k = n-k+1, a_{k+1}, \dots, a_n)$. Moreover, if $2k \geq n$ then from (2.2) and (2.3) we have the following result.

Proposition 3.1. *If $2k \geq n$ and the components are exchangeable, then the minimal signature of the $(C, k, n : G)$ system is $\boldsymbol{a} = (0, \dots, 0, a_k = n-k+1, a_{k+1} = k-n, 0, \dots, 0)$ and the maximal signature of the $(C, k, n : F)$ system is $\boldsymbol{b} = (0, \dots, 0, b_k = n-k+1, b_{k+1} = k-n, 0, \dots, 0)$.*

Obviously, the minimal signature of a system is equal to the maximal signature of its dual system. Hence, for $2k \geq n$ the reliability of the $(C, k, n : G)$ system can be computed by using the maximal signature of the $(C, k, n : F)$ system. Analogously, for $2k \geq n$ the reliability of the $(C, k, n : F)$ system can be computed by using the minimal signature of the $(C, k, n : G)$ system. In a similar way, from (3.1) the minimal signature of a parallel system is

$$\boldsymbol{a} = \left(\binom{n}{1}, -\binom{n}{2}, \binom{n}{3}, \dots, (-1)^{n+1}\binom{n}{n}\right). \tag{3.4}$$

Analogously, the maximal signature of a series system is

$$\boldsymbol{b} = \left(\binom{n}{1}, -\binom{n}{2}, \binom{n}{3}, \dots, (-1)^{n+1}\binom{n}{n}\right).$$

In general, the minimal and maximal signatures of k-out-of-n systems can be obtained from expressions (3.4.3) and (3.4.3$'$) of David and Nagaraja (2003, p. 46).

The minimal and maximal signatures of k-out-of-n and consecutive-k-out-of-n systems with $n = 3$ and $n = 4$ exchangeable components are given in Table 2. This table shows that Proposition 3.1 is not true for $2k < n$. For example, the minimal signature of the $(C, 1, 3 : G)$ system (a parallel system) is $\boldsymbol{a} = (3, -3, 1)$.

Note that minimal and maximal signatures allow us to write the distributions of coherent systems as negative mixtures (i.e. mixtures with some negative weights) of series or parallel systems. In the following lemma we give some properties (see Navarro and Hernandez (2005)) of these kinds of mixtures. We then use them to obtain new properties of consecutive-k-out-of-n systems.

TABLE 2: Minimal and maximal signatures for k-out-of-n and consecutive-k-out-of-n systems with $n = 3$ and $n = 4$ exchangeable components.

n	System	T	$\boldsymbol{a}$	$\boldsymbol{b}$
3	Series	$T_{1:3} = \min(T_1, T_2, T_3)$	(0, 0, 1)	(3, −3, 1)
3	$(C, 2, 3 : G)$	$T_{2\,\vert\,3:G} = \min(T_2, \max(T_1, T_3))$	(0, 2, −1)	(1, 1, −1)
3	2-out-of-3	$T_{2:3} = \max_{1\le i<j\le 3} \min(T_i, T_j)$	(0, 3, −2)	(0, 3, −2)
3	$(C, 2, 3 : F)$	$T_{2\,\vert\,3:F} = \max(T_2, \min(T_1, T_3))$	(1, 1, −1)	(0, 2, −1)
3	Parallel	$T_{3:3} = \max(T_1, T_2, T_3)$	(3, −3, 1)	(0, 0, 1)
4	Series	$T_{1:4} = \min(T_1, T_2, T_3, T_4)$	(0, 0, 0, 1)	(4, −6, 4, −1)
4	$(C, 3, 4 : G)$	$\max_{1\le i\le 2} \min(T_i, T_{i+1}, T_{i+2})$	(0, 0, 2, −1)	(2, 0, −2, 1)
4	3-out-of-4	$T_{3:4}$	(0, 0, 4, −3)	(0, 6, −8, 3)
4	$(C, 2, 4 : G)$	$\max_{1\le i\le 3} \min(T_i, T_{i+1})$	(0, 3, −2, 0)	(0, 3, −2, 0)
4	$(C, 2, 4 : F)$	$\min_{1\le i\le 3} \max(T_i, T_{i+1})$	(0, 3, −2, 0)	(0, 3, −2, 0)
4	2-out-of-4	$T_{2:4}$	(0, 6, −8, 3)	(0, 0, 4, −3)
4	$(C, 3, 4 : F)$	$\min_{1\le i\le 2} \max(T_i, T_{i+1}, T_{i+2})$	(2, 0, −2, 1)	(0, 0, 2, −1)
4	Parallel	$\max(T_1, T_2, T_3, T_4)$	(4, −6, 4, −1)	(0, 0, 0, 1)

Lemma 3.1. (Navarro and Hernandez (2005).) *If F_p are distribution functions such that $F_p = pF_1 + (1-p)F_0$ for $0 \le p \le p_{\max}$, where $1 \le p_{\max} \le \infty$ and $F_1 \ge_{\mathrm{mrl}} F_0$, $F_1 \ge_{\mathrm{hr}} F_0$, or $F_1 \ge_{\mathrm{st}} F_0$, then $F_p \le_{\mathrm{mrl}} F_{p'}$, $F_p \le_{\mathrm{hr}} F_{p'}$, or $F_p \le_{\mathrm{st}} F_{p'}$, respectively, for $0 \le p \le p' \le p_{\max}$.*

The proof can be obtained from http://www.um.es/docencia/jorgenav.

From this lemma and Proposition 3.1, we obtain the following results.

Proposition 3.2. *For $2k \ge n$, if $T_{1:k} \ge_{\mathrm{mrl}} T_{1:k+1}$ or $T_{1:k} \ge_{\mathrm{hr}} T_{1:k+1}$ then the lifetimes of $(C, k, n : G)$ systems with exchangeable components are respectively mrl-increasing or hr-increasing in n and respectively mrl-better or hr-better than the series system with k components, i.e.*

$$T_{1:k} \le_{\mathrm{mrl}} T_{k\,|\,n:G} \le_{\mathrm{mrl}} T_{k\,|\,n+1:G} \quad \textit{or, respectively,} \quad T_{1:k} \le_{\mathrm{hr}} T_{k\,|\,n:G} \le_{\mathrm{hr}} T_{k\,|\,n+1:G}.$$

Proposition 3.3. *For $2k \ge n$, if $T_{k:k} \le_{\mathrm{mrl}} T_{k+1:k+1}$ or $T_{k:k} \le_{\mathrm{hr}} T_{k+1:k+1}$ then the lifetimes of $(C, k, n : F)$ systems with exchangeable components are respectively mrl-decreasing or hr-decreasing in n and respectively mrl-worse or hr-worse than the parallel system with k components, i.e.*

$$T_{k:k} \ge_{\mathrm{mrl}} T_{k\,|\,n:F} \ge_{\mathrm{mrl}} T_{k\,|\,n+1:F} \quad \textit{or, respectively,} \quad T_{k:k} \ge_{\mathrm{hr}} T_{k\,|\,n:F} \ge_{\mathrm{hr}} T_{k\,|\,n+1:F}.$$

Navarro and Shaked (2006) proved that $T_{1:k} \ge_{\mathrm{hr}} T_{1:k+1}$ and $T_{k:k} \le_{\mathrm{hr}} T_{k+1:k+1}$ are in general not true for exchangeable distributions. They gave some conditions sufficient for these properties to hold. In the next example we show that in general $T_{1:k} \ge_{\mathrm{mrl}} T_{1:k+1}$ or $T_{k:k} \le_{\mathrm{mrl}} T_{k+1:k+1}$ need not always hold.

Example 3.1. Let (T_1, T_2) be a nonnegative random vector having the exchangeable density function described in Figure 3. A straightforward computation shows that the marginal density, $f_1(t)$, of T_1 is given by

$$f_1(t) = \begin{cases} (2t+1)\mathrm{e}^{-2t}, & 0 \le t < 1, \\ \frac{35}{4}\mathrm{e}^{-(10t-8)} + \frac{9}{4}\mathrm{e}^{-2t}, & t \ge 1, \end{cases}$$

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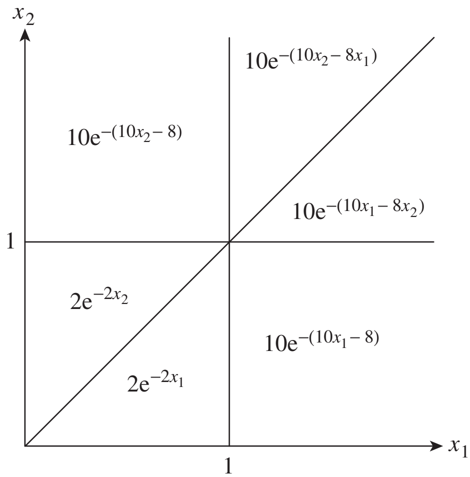

FIGURE 3: Density for Example 3.1.

and that the reliability function, $R_1(t)$, of T_1 is given by

$$R_1(t) = \begin{cases} (t+1)e^{-2t}, & 0 \le t < 1, \\ \frac{7}{8}e^{-(10t-8)} + \frac{9}{8}e^{-2t}, & t \ge 1. \end{cases}$$

Thus, the MRL function, $m_1(t)$, of T_1 is given by

$$m_1(t) = \begin{cases} \dfrac{1}{20}\dfrac{-12e^{-2} + 15e^{-2t} + 10te^{-2t}}{1+t}e^{2t}, & 0 \le t < 1, \\ \dfrac{1}{10}\dfrac{7e^{-10t+8} + 45e^{-2t}}{7e^{-10t+8} + 9e^{-2t}}, & t \ge 1. \end{cases}$$

Another lengthy but straightforward computation shows that the MRL function, $m_{1:2}(t)$, of the series system $T_{1:2} = \min\{T_1, T_2\}$ is given by

$$m_{1:2}(t) = 0.5, \qquad t \ge 0.$$

Since $m_1(0) = 0.6688 > 0.5$ and $m_1(1) = 0.325 < 0.5$, T_1 and $T_{1:2}$ are not mrl-ordered. Analogously, the MRL function, $m_{2:2}(t)$, of the parallel system $T_{2:2} = \max(T_1, T_2)$ satisfies $m_{2:2}(0) = 0.8376 > 0.6688 = m_1(0)$ and $m_{2:2}(1) = 0.26667 < 0.325 = m_1(1)$; hence, $T_{2:2}$ and T_1 are not mrl-ordered. The MRL functions can be seen in Figure 4.

If the components are i.i.d., then $T_{1:k} \ge_{\mathrm{lr}} T_{1:k+1}$, $T_{1:k} \ge_{\mathrm{hr}} T_{1:k+1}$, $T_{1:k} \ge_{\mathrm{mrl}} T_{1:k+1}$, $T_{k:k} \le_{\mathrm{lr}} T_{k+1:k+1}$, $T_{k:k} \le_{\mathrm{hr}} T_{k+1:k+1}$, and $T_{k:k} \le_{\mathrm{mrl}} T_{k+1:k+1}$ hold (see Shaked and Shanthikumar (1994, p. 37)). Therefore, we have the following corollary, which extends the analogous result on the stochastic ordering given by Boland and Samaniego (2004).

Corollary 3.1. *If the components are i.i.d., then*

$$T_{k\,|\,n:G} \le_{\mathrm{hr}} T_{k\,|\,n+1:G}, \qquad T_{k\,|\,n:G} \le_{\mathrm{mrl}} T_{k\,|\,n+1:G},$$

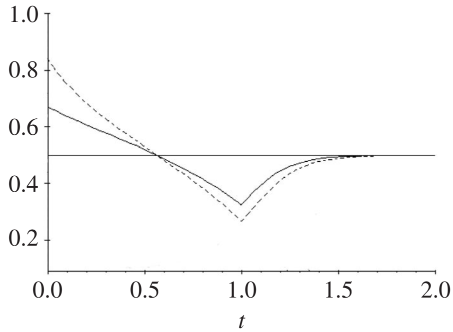

FIGURE 4: MRL functions of the component system (*nonconstant continuous line*), the parallel system (*dashed line*), and the series system (*constant continuous line*) in Example 3.1.

and

$$T_{k\,|\,n:F} \geq_{\mathrm{hr}} T_{k\,|\,n+1:F}, \qquad T_{k\,|\,n:F} \geq_{\mathrm{mrl}} T_{k\,|\,n+1:F},$$

for $k \geq n/2$.

Note that if $T_{k\,|\,n:G} \leq_{\mathrm{mrl}} T_{1:k-1}$ or $T_{k\,|\,n:G} \leq_{\mathrm{hr}} T_{1:k-1}$ holds, then $T_{k\,|\,n:G} \leq_{\mathrm{mrl}} T_{k-1\,|\,n:G}$ or, respectively, $T_{k\,|\,n:G} \leq_{\mathrm{hr}} T_{k-1\,|\,n:G}$ holds. However, the following example shows that $T_{k\,|\,n:G} \leq_{\mathrm{mrl}} T_{1:k-1}$ and $T_{k\,|\,n:G} \leq_{\mathrm{hr}} T_{1:k-1}$ in general do not hold.

Example 3.2. If the components in a system are i.i.d. with exponential distributions and common reliability function $R(t) = \exp(-\lambda t)$ for $t \geq 0$, where $\lambda > 0$, then $m_{1:k}(t) = 1/(k\lambda)$. Therefore, for $k \geq n/2$, $m_{k\,|\,n:G}(0) = (n+1)/(k(k+1)\lambda)$ and, so, $m_{k\,|\,n:G}(0) > m_{1:k-1}(0)$ for $n = 6$ and $k = 3$.

We shall use the following lemma to study the asymptotic behaviour of the MRL function of consecutive-k-out-of-n systems.

Lemma 3.2. (Navarro and Hernandez (2005).) *Let $F(t)$ be a CDF such that*

$$F(t) = \sum_{i=1}^{n} p_i F_i(t)$$

for $t \geq 0$, where $F_1(t), F_2(t), \ldots, F_n(t)$ are distribution functions such that $F_i(t) < 1$ for all t and $p_1, p_2, \ldots, p_n$ are real numbers such that $p_i \neq 0$ and $\sum_{i=1}^{n} p_i = 1$. Let $m(t)$ be the MRL function of $F(t)$ and let $m_i(t)$ be the MRL function of $F_i(t)$, $i = 1, 2, \ldots, n$. If

$$\liminf_{t\to\infty} \frac{m_1(t)}{m_i(t)} > 1 \quad \textit{and} \quad \limsup_{t\to\infty} \frac{m_1(t)}{m_i(t)} < \infty$$

for $i = 2, 3, \ldots, n$, then $p_1 > 0$ and

$$\lim_{t\to\infty} \frac{m(t)}{m_1(t)} = 1.$$

The proof can be obtained from http://www.um.es/docencia/jorgenav.

If the systems have exchangeable components, then the preceding lemma can be applied to the representations as mixtures of distributions of consecutive-k-out-of-n systems by using Samaniego minimal or maximal signatures. Actually, in the general case it can also be applied to the representation (3.1). However, the best results are obtained by using the minimal signature. For example, if $2k \geq n$ then from Proposition 3.1 we obtain the following result.

Proposition 3.4. *If the* $(C, k, n : G)$ *system has exchangeable components,* $2k \geq n$*,*

$$\liminf_{t\to\infty} \frac{m_{1:k}(t)}{m_{1:k+1}(t)} > 1, \quad \textit{and} \quad \limsup_{t\to\infty} \frac{m_{1:k}(t)}{m_{1:k+1}(t)} < \infty,$$

then

$$\lim_{t\to\infty} \frac{m_{k\,|\,n:G}(t)}{m_{1:k}(t)} = 1.$$

A similar result can be obtained by using (3.3). The following example shows how this result can be applied to systems with exchangeable dependent components.

Example 3.3. If the lifetimes, $T_1, T_2, \dots, T_n$, of the components of a system have the Farlie–Gumbel–Morgenstern distribution with standard exponential marginals, that is, they have the joint reliability function

$$R(t_1, t_2, \dots, t_n) = \left(1+\alpha \prod_{i=1}^{n}(1-\mathrm{e}^{-t_i})\right) \exp\left(-\sum_{i=1}^{n} t_i\right), \qquad (t_1, t_2, \dots, t_n) \geq (0, 0, \dots, 0),$$

where $|\alpha| \leq 1$, then it is not hard to verify that, for $j < n$, every j-dimensional marginal distribution of $R(t)$ is the joint distribution of j independent standard exponential random variables. Therefore,

$$m_{1:j}(t) = 1/j, \qquad j = 1, 2, \dots, n-1.$$

Moreover, a straightforward computation yields

$$\lim_{t\to\infty} m_{1:n}(t) = 1/n.$$

Therefore,

$$\lim_{t\to\infty} \frac{m_{1:k}(t)}{m_{1:k+1}(t)} = \frac{k+1}{k}$$

for $k = 1, 2, \dots, n$. Thus, Lemma 3.2 applies to representation (3.3) of the $(C, k, n : G)$ system and, so,

$$\lim_{t\to\infty} m_{k\,|\,n:G}(t) = 1/k$$

for $k = 1, 2, \dots, n-1$, that is, $m_{k\,|\,n:G}(t)$ is asymptotically equivalent to $m_{1:k}(t) = 1/k$ for $k = 1, 2, \dots, n-1$. Therefore, $(C, k, n : G)$ systems are strictly mrl-ordered in k as $t \to \infty$, for $k = 1, 2, \dots, n$.

The next proposition gives a condition sufficient to have a similar property in the case of i.i.d. components.

Proposition 3.5. *If the components in a system are i.i.d. with common hazard function* $h(t)$ *such that* $\lim_{t\to\infty} h(t) = \lambda$*,* $0 < \lambda < \infty$*, then*

$$\lim_{t\to\infty} \frac{m_{1:k}(t)}{m_{1:k+1}(t)} = \frac{k+1}{k} \tag{3.5}$$

and

$$\lim_{t\to\infty} m_{k\,|\,n:G}(t) = \frac{1}{k\lambda}$$

for $k = 1, 2, \dots, n$.

Proof. Let $m(t)$ be the MRL function of the components. Then, from (2.1) and l'Hôpital's rule, we have

$$\lim_{t\to\infty} m(t) = \lim_{t\to\infty} \frac{1}{h(t)} = \frac{1}{\lambda}.$$

Moreover, if the components are i.i.d., then the hazard function of the series system with k components is $h_{1:k}(t) = kh(t)$. Therefore, if $0 < \lambda < \infty$ then

$$\lim_{t\to\infty} m_{1:k}(t) = \lim_{t\to\infty} \frac{1}{h_{1:k}(t)} = \frac{1}{k\lambda}$$

for $k = 1, 2, \dots, n$, and by using (3.3) and Lemma 3.2 we obtain the stated result.

In the conditions of the preceding proposition, there must exist a $t_0 > 0$ such that $m_{k\,|\,n:G}(t) > m_{k+1\,|\,n:G}(t)$ for all $t > t_0$ (i.e. they are asymptotically ordered). It is easy to show that if the components are independent, then $h_{1:k}(t)/h_{1:k+1}(t) = k/(k+1)$. The following example shows that (3.5) is not necessarily true.

Example 3.4. If the components in a system are i.i.d. with reliability function $R(t) = b^{1/a}(at+b)^{-1/a}$ for $t \ge 0$ and $a, b > 0$ (i.e. they are Pareto type II), then a straightforward computation yields $m_{1:k}(t) = (at+b)/(k-a)$ for $k = 1, 2, \dots, n$. Therefore, $m_{1:k}(t)/m_{1:k+1}(t) = (k+1-a)/(k-a)$ and (3.5) does hold. However, $(k+1-a)/(k-a) > 1$ and, hence, Proposition 3.4 can be applied to the MRL function of the $(C, k, n : G)$ system, yielding

$$\lim_{t\to\infty} \frac{m_{k\,|\,n:G}(t)}{at+b} = \frac{1}{k-a}$$

for $k \ge n/2$. Thus, $(C, k, n : G)$ systems are asymptotically strictly mrl-ordered in k for $k \ge n/2$.

Proposition 3.6. *If the components in a system are i.i.d. with common hazard function* $h(t)$ *such that* $\lim_{t\to\infty} h(t) = \lambda$, $0 < \lambda < \infty$, *and if* s *is an integer such that* $n/(s+1) < k \le n/s$, *then*

$$\lim_{t\to\infty} m_{k\,|\,n:F}(t) = \frac{1}{s\lambda}.$$

Proof. The minimal cut sets of the $(C, k, n : F)$ system are $C_1 = \{1, 2, \dots, k\}$, $C_2 = \{2, \dots, k+1\}$, $\dots$, $C_{n-k+1} = \{n-k+1, n-k+2, \dots, n\}$. Therefore, if $n/(s+1) < k \le n/s$ then $\{k, 2k, \dots, sk\}$ is a minimal path set of the $(C, k, n : F)$ system. Moreover, the minimal path sets have at least s elements since $C_1, C_{k+1}, \dots, C_{(s-1)k+1}$ are disjoint minimal cut sets and, for $j = 1, k+1, \dots, (s-1)k+1$, every minimal path set must have an element in every C_j. Therefore, the minimal signature of the $(C, k, n : F)$ system is $\boldsymbol{a} = (0, \dots, 0, a_s, a_{s+1}, \dots, a_n)$, where $a_s > 0$. The rest of the proof is thus similar to that of Proposition 3.5.

Next we obtain some properties concerning the monotonicity of the MRL functions of consecutive-k-out-of-n systems, based on the following lemma.

Lemma 3.3. (Navarro and Hernandez (2005).) *Let $F(t)$, $F_1(t)$, and $F_2(t)$ be differentiable distribution functions such that $F(t) = w_1 F_1(t) + w_2 F_2(t)$ for all t. Then*

$$\frac{m'(t)}{m^2(t)} = \alpha(t)\frac{m_1'(t)}{m_1^2(t)} + (1-\alpha(t))\frac{m_2'(t)}{m_2^2(t)} + \alpha(t)(1-\alpha(t))\frac{(m_1(t)-m_2(t))^2}{m_1^2(t)m_2^2(t)},$$

where $\alpha(t) = w_1 G_1(t)/G(t)$, $G(t) = R(t)m(t)$, and $G_1(t) = R_1(t)m_1(t)$.

The proof can be obtained from http://www.um.es/docencia/jorgenav.

In particular, if $0 < w_1 < 1$ and $F_1(t)$ and $F_2(t)$ are IMRL (i.e. increasing MRL in a wide sense) distributions, then $F(t)$ is also IMRL (a well-known property of positive mixtures). It is also well known that the positive mixtures of DMRL (i.e. decreasing MRL in a wide sense) distributions are not DMRL (see, e.g. Wondmagegnehu *et al.* (2005)). Analogously, if $w_1 > 1$, $F_1(t)$ is DMRL, and $F_2(t)$ is IMRL, then $F(t)$ is DMRL. As immediate consequences we have the following properties.

Proposition 3.7. *If the $(C, k, n : G)$ system has exchangeable components, $2k \geq n$, $T_{1:k}$ is DMRL, and $T_{1:k+1}$ is IMRL, then $T_{k\,|\,n:G}$ is DMRL.*

Proposition 3.8. *If the $(C, k, n : F)$ system has exchangeable components, $2k \geq n$, $T_{k:k}$ is DMRL, and $T_{k+1:k+1}$ is IMRL, then $T_{k\,|\,n:F}$ is DMRL.*

Example 3.5. In the common case of systems with independent, exponential components of mean μ, we see that the series systems are also exponential, with mean $\mu_{1:k} = \mathrm{E}(T_{1:k}) = \mu/k$ for $k = 1, 2, \dots, n$, and hence are both IMRL and DMRL. Therefore, the $(C, k, n : G)$ systems are DMRL for $k \geq n/2$. Figure 1 shows that this result is in general not true for $k < n/2$ (e.g. for $(C, k, n : F)$ systems with $k \geq n/2$). Moreover, from Proposition 3.2, $T_{k\,|\,n:G} \geq_{\mathrm{mrl}} T_{1:k}$ holds. Hence, if

$$m_{k\,|\,n:G}(0) = \frac{n+1}{k(k+1)}\mu \leq m_{1:k-1}(0) = \frac{1}{(k-1)}\mu,$$

then

$$T_{k\,|\,n:G} \leq_{\mathrm{mrl}} T_{k-1\,|\,n:G}$$

for $2(k-1) \geq n$. Finally, we note that in this case Lemma 3.2 cannot be applied to the representation of the $(C, k, n : F)$ systems in terms of the maximal signature since the MRL functions of the parallel systems, from (3.4), satisfy $\lim_{t\to\infty} m_{k:k}(t) = \mu$ and

$$\mu_{k:k} = m_{k:k}(0) = \sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}\frac{\mu}{i}$$

for $k = 1, 2, \dots, n$, where $\mu_{k:k} = \mathrm{E}(T_{k:k})$. Therefore, for $k \geq n/2$,

$$F_{k\,|\,n:F}(t) = (n-k+1)\sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}F_{1:i}(t) - (n-k)\sum_{i=1}^{k+1}(-1)^{i+1}\binom{k+1}{i}F_{1:i}(t)$$

and

$$\mu_{k\,|\,n:F} = m_{k\,|\,n:F}(0) = (n-k+1)\sum_{i=1}^{k}(-1)^{i+1}\binom{k}{i}\frac{\mu}{i} - (n-k)\sum_{i=1}^{k+1}(-1)^{i+1}\binom{k+1}{i}\frac{\mu}{i},$$

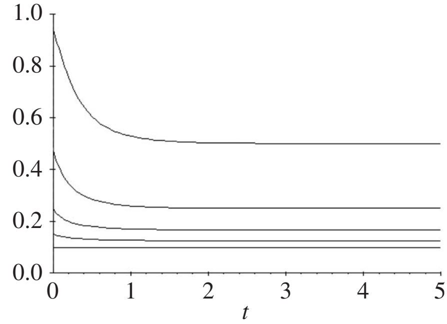

FIGURE 5: MRL functions of $(C, i, 5 : G)$ systems, $i = 1, 2, \dots, 5$ (in order from top to bottom), with exponential components with $\lambda = 2$.

where $\mu_{k\,|\,n:F} = \mathrm{E}(T_{k\,|\,n:F})$. Hence, $\lim_{t\to\infty} m_{k\,|\,n:F}(t) = \mu$ for $k \geq n/2$, that is, for $k \geq n/2$ the consecutive-k-out-of-$n : F$ systems are asymptotically equivalent in the mrl order (see Figures 1 and 2). Figure 5 shows that in this case the consecutive-k-out-of-$n : G$ systems are mrl-ordered for $\lambda = 2$, $k = 1, 2, \dots, 5$, and $n = 5$.

Finally, we obtain the main result of this section, showing that the consecutive-k-out-of-n systems with i.i.d. components are lr-, mrl-, and hr-ordered in k for $2k \geq n$. First, we need a lemma.

Lemma 3.4. *For $2k \geq n$ and $1 \leq k \leq n$, the function*

$$g(x) = \frac{(n-k)(k+1)x - (n-k-1)(k+2)x^2}{(n-k+1)k - (n-k)(k+1)x}$$

is increasing in x for $0 \leq x \leq 1$.

The proof can be obtained from http://www.um.es/docencia/jorgenav.

Theorem 3.1. *If the components are i.i.d., then*

$$T_{k\,|\,n:G} \geq_{\mathrm{lr}} T_{k+1\,|\,n:G} \quad \text{and} \quad T_{k\,|\,n:F} \leq_{\mathrm{lr}} T_{k+1\,|\,n:F}$$

for $n/2 \leq k < n-1$.

Proof. Let $f_{k\,|\,n:G}$ denote the density function of $T_{k\,|\,n:G}$. For $2k \geq n$, from (2.3) we have

$$\frac{f_{k+1\,|\,n:G}(t)}{f_{k\,|\,n:G}(t)} = \frac{(n-k)(k+1)R(t) - (n-k-1)(k+2)R^2(t)}{(n-k+1)k - (n-k)(k+1)R(t)},$$

where $R(t)$ is the common reliability function of the components. As $R(t)$ is decreasing and $0 \leq R(t) \leq 1$, from Lemma 3.4 $f_{k+1\,|\,n:G}(t)/f_{k\,|\,n:G}(t)$ is decreasing in t and, hence,

$$T_{k\,|\,n:G} \geq_{\mathrm{lr}} T_{k+1\,|\,n:G}$$

holds for $k \geq n/2$.

Analogously, for $2k \geq n$, from (2.2) we have

$$\frac{f_{k+1\,|\,n:F}(t)}{f_{k\,|\,n:F}(t)} = \frac{(n-k)(k+1)F(t) - (n-k-1)(k+2)F^2(t)}{(n-k+1)k - (n-k)(k+1)F(t)},$$

where $F(t)$ is the common distribution function of the components. As $F(t)$ is increasing and $0 \leq F(t) \leq 1$, from Lemma 3.4 $f_{k+1\,|\,n:F}(t)/f_{k\,|\,n:F}(t)$ is increasing and, hence,

$$T_{k\,|\,n:F} \leq_{\mathrm{lr}} T_{k+1\,|\,n:F}$$

holds for $k \geq n/2$.

The authors do not know if this property holds for consecutive-k-out-of-n systems with $2k < n$. Note that Example 3.1 shows that this property is in general not true for systems with exchangeable components.

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