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EULER NUMBERS MODULO 2^n

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Abstract

Let $\{E_n\}$ be the Euler numbers. We give a general congruence modulo $2^{(m+2)n}$ for $E_{2^m k+b}$, where k, m, n are positive integers and $b \in \{0, 2, 4, \ldots\}$.

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1. Introduction

The Euler numbers $\{E_n\}$ are given by

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} \quad \left(|t| < \frac{\pi}{2}\right).$$

It is well known that

$$E_0 = 1, \quad E_{2n-1} = 0, \quad \sum_{r=0}^{n} \binom{2n}{2r} E_{2r} = 0 \ (n = 1, 2, 3, \ldots).$$

The first few Euler numbers are shown below:

$$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385, \quad E_{10} = -50\,521,$$
$$E_{12} = 2\,702\,765, \quad E_{14} = -199\,360\,981, \quad E_{16} = 19\,391\,512\,145.$$

Let $\mathbb{N}$ be the set of positive integers. For $m \in \mathbb{N}$ and $b \in \{0, 2, 4, \ldots\}$, in 1875 Stern (see [1, 6]) showed that

$$E_{2^m+b} \equiv 2^m + E_b \mod 2^{m+1}. \tag{1.1}$$

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Let $[x]$ be the greatest integer not exceeding x. In 2005, Sun [4] showed that for any $k, n \in \mathbb{N}$,

$$\frac{3^{2k+1}+1}{4}E_{2k} \equiv \frac{3^{2k}}{2}\sum_{j=0}^{2^n-1}(-1)^{j-1}(2j+1)^{2k}\left[\frac{3j+1}{2^n}\right] \mod 2^n \tag{1.2}$$

and deduced (1.1) from (1.2).

For $k, m, n \in \mathbb{N}$ and $b \in \{0, 2, 4, \ldots\}$, in [5, Corollary 7.4] the author showed that

$$E_{2^m k+b} \equiv \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}E_{2^m r+b} \mod 2^{mn+n-\alpha}, \tag{1.3}$$

where $\alpha \in \mathbb{N}$ is given by $2^{\alpha-1} \le n < 2^\alpha$.

In Section 2, we give several basic lemmas. In Section 3, using (1.2) and some results in [2, 3, 5], we deduce a general congruence modulo $2^{(m+2)n}$ for $E_{2^m k+b}$, where $b, k \in \{0, 1, 2, \ldots\}$, $2 \mid b$ and $n \in \mathbb{N}$ (see Theorem 3.3). As a consequence we show that for $m \ge 4$,

$$E_{2^m k+b} \equiv \begin{cases} E_b + 5\cdot 2^m k \mod 2^{m+4} & \text{if } b \equiv 0, 6 \mod 8, \\ E_b - 3\cdot 2^m k \mod 2^{m+4} & \text{if } b \equiv 2, 4 \mod 8 \end{cases}$$

and

$$E_{4k} \equiv \frac{1+688k-384k^2}{1-1316k-1184k^2-1536k^3} \mod 4096.$$

2. Basic lemmas

LEMMA 2.1. *For $n \in \mathbb{N}$,*

$$\sum_{j=0}^{2^n-1}(-1)^{j-1}\left[\frac{3j+1}{2^n}\right] = 2.$$

PROOF. For $j \in \{0, 1, \ldots, 2^n-1\}$,

$$\left[\frac{3j+1}{2^n}\right] = \begin{cases} 0 & \text{if } j < \dfrac{2^n-1}{3}, \\ 1 & \text{if } \dfrac{2^n-1}{3} \le j < \dfrac{2^{n+1}-1}{3}, \\ 2 & \text{if } \dfrac{2^{n+1}-1}{3} \le j \le 2^n-1. \end{cases} \tag{2.1}$$

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Thus

$$\sum_{j=0}^{2^n-1}(-1)^{j-1}\left[\frac{3j+1}{2^n}\right]=\begin{cases}\displaystyle\sum_{j=(2^n-1)/3}^{(2^{n+1}-2)/3}(-1)^{j-1}+2\sum_{j=(2^{n+1}+1)/3}^{2^n-1}(-1)^{j-1} & \text{if } 2\mid n,\\ \displaystyle\sum_{j=(2^n+1)/3}^{(2^{n+1}-4)/3}(-1)^{j-1}+2\sum_{j=(2^{n+1}-1)/3}^{2^n-1}(-1)^{j-1} & \text{if } 2\nmid n\end{cases}$$
$$=0+2=2.$$ □

LEMMA 2.2. *Let* $b\in\{0,2,4,\ldots\}$ *and* $n\in\mathbb{N}$. *Then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(3+3^{-2k-b})E_{2k+b}\equiv 0 \mod 2^{3n+2}.$$

PROOF. By Lemma 2.1 we know that (1.2) is also true for $k=0$ and $n\in\mathbb{N}$. Thus, replacing n with $3n$ and k with $k+b/2$ in (1.2), we see that for $k\in\{0,1,2,\ldots\}$ and $b\in\{0,2,4,\ldots\}$,

$$(3+3^{-2k-b})E_{2k+b}\equiv 2\sum_{j=0}^{2^{3n}-1}(-1)^{j-1}(2j+1)^{2k+b}\left[\frac{3j+1}{2^{3n}}\right] \mod 2^{3n+2}.$$

Therefore,

$$\begin{aligned}&\sum_{k=0}^{n}\binom{n}{k}(-1)^k(3+3^{-2k-b})E_{2k+b}\\
&\equiv 2\sum_{j=0}^{2^{3n}-1}(-1)^{j-1}\left[\frac{3j+1}{2^{3n}}\right]\sum_{k=0}^{n}\binom{n}{k}(-1)^k(2j+1)^{2k+b}\\
&=2\sum_{j=0}^{2^{3n}-1}(-1)^{j-1}\left[\frac{3j+1}{2^{3n}}\right](1-(2j+1)^2)^n(2j+1)^b\\
&=2^{3n+1}\sum_{j=0}^{2^{3n}-1}(-1)^{j-1}\left[\frac{3j+1}{2^{3n}}\right]\left(\frac{1-(2j+1)^2}{8}\right)^n(2j+1)^b \mod 2^{3n+2}.\end{aligned}$$

Since

$$\left(\frac{1-(2j+1)^2}{8}\right)^n\equiv\frac{1-(2j+1)^2}{8}=-\frac{j(j+1)}{2}$$
$$\equiv\begin{cases}1 \mod 2 & \text{if } j\equiv 1,2 \mod 4,\\ 0 \mod 2 & \text{if } j\equiv 0,3 \mod 4,\end{cases}$$

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using (2.1) and the fact that $(2^{3n}-1)/3 \equiv (2^{3n+1}-1)/3 \equiv 3 \bmod 4$, we see that

$$\sum_{j=0}^{2^{3n}-1}(-1)^{j-1}\left[\frac{3j+1}{2^{3n}}\right]\left(\frac{1-(2j+1)^2}{8}\right)^n(2j+1)^b \equiv \sum_{\substack{((2^{3n}-1)/3)\le j<((2^{3n+1}-1)/3)\\ j\equiv 1,2 \bmod 4}}(-1)^{j-1}=0 \pmod 2.$$

Combining all of the above, we obtain the result. □

LEMMA 2.3. *Let* $b \in \{0, 2, 4, \ldots\}$ *and* $n \in \mathbb{N}$. *Then*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{3^{2k+b+1}+1}{4}E_{2k+b}\equiv 0 \pmod{2^{3n}}.$$

PROOF. Lemma 2.1 of [3] states that for any functions f and g,

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k)g(k) = \sum_{s=0}^{n}\binom{n}{s}\left(\sum_{r=0}^{s}\binom{s}{r}(-1)^r\sum_{j=0}^{n-s+r}\binom{n-s+r}{j}(-1)^j f(j)\right)\sum_{i=0}^{s}\binom{s}{i}(-1)^i g(i).$$

Set $f(k)=\frac{1}{4}(3+3^{-2k-b})E_{2k+b}$ and $g(k)=3^{2k+b}$. Then

$$\sum_{i=0}^{s}\binom{s}{i}(-1)^i g(i)=\sum_{i=0}^{s}\binom{s}{i}(-1)^i 3^{2i+b}=(1-9)^s 3^b\equiv 0 \pmod{2^{3s}}$$

and

$$\sum_{j=0}^{n-s+r}\binom{n-s+r}{j}(-1)^j f(j)\equiv 0 \pmod{2^{3(n-s+r)}}$$

by Lemma 2.2. As $2^{3(n-s+r)}\cdot 2^{3s}=2^{3n+3r}\equiv 0 \bmod 2^{3n}$, from the above we see that

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{3^{2k+b+1}+1}{4}E_{2k+b} &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k f(k)g(k)\\ &= \sum_{s=0}^{n}\binom{n}{s}\left(\sum_{r=0}^{s}\binom{s}{r}(-1)^r\sum_{j=0}^{n-s+r}\binom{n-s+r}{j}(-1)^j f(j)\right)\\ &\quad\times\sum_{i=0}^{s}\binom{s}{i}(-1)^i g(i)\\ &\equiv 0 \pmod{2^{3n}}.\end{aligned}$$

This proves the lemma. □

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3. Main results

THEOREM 3.1. *Let $b, k \in \{0, 1, 2, \ldots\}$, $2 \mid b$ and $m, n, t \in \mathbb{N}$. Then*

$$\frac{3^{2^m kt+b+1}+1}{4}E_{2^m kt+b} \equiv \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\frac{3^{2^m rt+b+1}+1}{4}E_{2^m rt+b} \mod 2^{(m+2)n}.$$

Moreover,

$$\frac{3^{2^m kt+b+1}+1}{4}E_{2^m kt+b} \equiv \sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\frac{3^{2^m rt+b+1}+1}{4}E_{2^m rt+b} + 2^{(m+2)n}\binom{k}{n}(-t)^n \frac{\sum_{r=0}^{n}\binom{n}{r}(-1)^r \frac{3^{2r+b+1}+1}{4}E_{2r+b}}{2^{3n}} \mod 2^{(m+2)n+2}.$$

PROOF. Let $f(k) = \frac{1}{4}(3^{2k+b+1}+1)E_{2k+b}$ and $A_k = 2^{-k}\sum_{r=0}^{k}\binom{k}{r}(-1)^r f(r)$. From Lemma 2.3 we know that $2^{2k} \mid A_k$. Putting $p = 2$ and $f(k) = \frac{1}{4}(3^{2k+b+1}+1)E_{2k+b}$ in a formula in [5, p. 88], we have

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^r f(2^{m-1}rt) = 2^{mn}t^n A_n + \sum_{k=n+1}^{2^{m-1}nt}(-2)^n(-1)^k A_k \times \sum_{j=n}^{k}(-1)^{k-j}\frac{s(k,j)j!}{k!}2^{k-j}\cdot\frac{S(j,n)n!}{j!}2^{j-n}\cdot(2^{m-1}t)^j,$$

where $s(n,k)$ and $S(n,k)$ are Stirling numbers given by

$$x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n}(-1)^{n-k}s(n,k)x^k$$

and

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)\cdots(x-k+1).$$

By [5, Lemma 4.2], for $j, k, n \in \mathbb{N}$, $(s(k,j)j!/k!)2^{k-j}$ and $(S(j,n)n!/j!)2^{j-n}$ are rational 2-integers. Thus, by the above and the fact that $2^{2k} \mid A_k$,

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^r f(2^{m-1}rt) \equiv 2^{(m+2)n}t^n\cdot\frac{A_n}{2^{2n}} \mod 2^{(m+2)n+2}.$$

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From [2, Lemma 2.1] we know that

$$f(2^{m-1}kt)=\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(2^{m-1}rt)\\ +\sum_{r=n}^{k}\binom{k}{r}(-1)^r\sum_{s=0}^{r}\binom{r}{s}(-1)^s f(2^{m-1}st).$$

For $r\ge n+1$ we have $(m+2)r\ge(m+2)(n+1)\ge(m+2)n+2$. Thus applying the above we obtain $\sum_{s=0}^{r}\binom{r}{s}(-1)^s f(2^{m-1}st)\equiv 0 \bmod 2^{(m+2)n+2}$ and so

$$f(2^{m-1}kt)\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}f(2^{m-1}rt)\\ +\binom{k}{n}(-t)^n\cdot 2^{(m+2)n}\cdot\frac{A_n}{2^{2n}} \mod 2^{(m+2)n+2}.$$

Thus the theorem is proved. □

Taking $m=t=1$ and $b=0$ in Theorem 3.1 leads to the following congruence.

COROLLARY 3.2. *For any nonnegative integer k and positive integer n,*

$$\frac{3^{2k+1}+1}{4}E_{2k}\equiv\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\frac{3^{2r+1}+1}{4}E_{2r} \mod 2^{3n}.$$

THEOREM 3.3. *Let* $b,k\in\{0,1,2,\ldots\}$, $2\mid b$ *and* $m,n,t\in\mathbb{N}$. *Then*

$$E_{2^mkt+b}\equiv\frac{\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\frac{3^{2^mrt+b+1}+1}{4}E_{2^mrt+b}}{\frac{3^{b+1}+1}{4}+3^{b+1}\sum_{r=1}^{n}\binom{k}{r}\frac{(3^{2^mt}-1)^r}{4}} \mod 2^{(m+2)n}.$$

PROOF. As $3^{2^{m+1}}-1=(3^{2^m}-1)(3^{2^m}+1)$, by induction we see $2^{m+2}\mid(3^{2^m}-1)$ and so $2^{m+2}\mid(3^{2^mt}-1)$. Thus

$$\begin{aligned}\frac14(3^{2^mkt+b+1}+1)&=\frac14(1+3^{b+1}(1+(3^{2^mt}-1))^k)\\ &=\frac{3^{b+1}+1}{4}+\frac{3^{b+1}}{4}\sum_{r=1}^{k}\binom{k}{r}(3^{2^mt}-1)^r\\ &\equiv\frac{3^{b+1}+1}{4}+\frac{3^{b+1}}{4}\sum_{r=1}^{n}\binom{k}{r}(3^{2^mt}-1)^r \mod 2^{(m+2)n}.\end{aligned}$$

Note that $\frac14(3^{2^mkt+b+1}+1)\equiv\frac14(3+1)=1 \bmod 2$. Applying the above and Theorem 3.1, we obtain the result. □

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COROLLARY 3.4. *For any nonnegative integer k and positive integer n,*

$$E_{2k} \equiv \frac{\displaystyle\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\frac{3^{2r+1}+1}{4}E_{2r}}{\displaystyle 1+3\sum_{r=1}^{n}\binom{k}{r}2^{3r-2}} \mod 2^{3n}.$$

PROOF. Putting $m = t = 1$ and $b = 0$ in Theorem 3.3, we deduce the result. □

Taking $n = 1, 2, 3$ in Corollary 3.4 we have the following corollary.

COROLLARY 3.5. *For any nonnegative integer k,*

$$E_{2k} \equiv \frac{1}{1-2k} \mod 8,$$
$$E_{2k} \equiv \frac{1-8k}{1-18k+24k^2} \mod 64,$$
$$E_{2k} \equiv \frac{1-168k+160k^2}{1+110k-168k^2+64k^3} \mod 512.$$

We note that different congruences for E_{2k} mod 64 and E_{2k} mod 256 have been given by the author in [5, p. 111].

COROLLARY 3.6. *For any $k \in \{0, 1, 2, \ldots\}$ and $n \in \mathbb{N}$,*

$$E_{4k} \equiv \frac{\displaystyle\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\frac{3^{4r+1}+1}{4}E_{4r}}{\displaystyle 1+3\sum_{r=1}^{n}\binom{k}{r}4^{2r-1}5^{r}} \mod 2^{4n}$$

and

$$E_{4k+2} \equiv \frac{\displaystyle\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\frac{3^{4r+3}+1}{4}E_{4r+2}}{\displaystyle 7+27\sum_{r=1}^{n}\binom{k}{r}4^{2r-1}5^{r}} \mod 2^{4n}.$$

PROOF. Putting $m = 2$, $t = 1$ and $b = 0, 2$ in Theorem 3.3, we deduce the result. □

Taking $n = 3$ in Corollary 3.6 we deduce the following result.

COROLLARY 3.7. *For any nonnegative integer k,*

$$E_{4k} \equiv \frac{1+688k-384k^2}{1-1316k-1184k^2-1536k^3} \mod 4096$$

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and

$$E_{4k+2} \equiv \frac{-7-1232k+640k^2}{7+444k+1632k^2-1536k^3} \pmod{4096}.$$

Putting $m=3$ and $t=1$ in Theorem 3.3 leads to the following corollary.

COROLLARY 3.8. *Let $k \in \{0, 1, 2, \ldots\}$, $b \in \{0, 2, 4, \ldots\}$ and $n \in \mathbb{N}$. Then*

$$E_{8k+b} \equiv \frac{\displaystyle\sum_{r=0}^{n-1}(-1)^{n-1-r}\binom{k-1-r}{n-1-r}\binom{k}{r}\frac{3^{8r+b+1}+1}{4}E_{8r+b}}{\displaystyle\frac{3^{b+1}+1}{4}+3^{b+1}\sum_{r=1}^{n}\binom{k}{r}2^{5r-2}205^r} \pmod{2^{5n}}.$$

THEOREM 3.9. *Let $b \in \{0, 2, 4, \ldots\}$ and $k, m \in \mathbb{N}$ with $m \geq 4$. Then*

$$E_{2^m k+b} \equiv \begin{cases} E_b + 5\cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 0, 6 \pmod 8, \\ E_b - 3\cdot 2^m k \pmod{2^{m+4}} & \text{if } b \equiv 2, 4 \pmod 8. \end{cases}$$

Furthermore,

$$E_{8k+b} \equiv \begin{cases} E_b - 24k \pmod{128} & \text{if } b \equiv 0, 6 \pmod 8, \\ E_b + 40k \pmod{128} & \text{if } b \equiv 2, 4 \pmod 8. \end{cases}$$

PROOF. Suppose that $m \geq 3$. Taking $n=t=1$ in Theorem 3.1, we see that

$$\begin{aligned} &\frac{3^{2^m k+b+1}+1}{4}E_{2^m k+b} \\ &\quad\equiv \frac{3^{b+1}+1}{4}E_b - 2^{m+2}k\cdot\frac{1}{8}\left(\frac{3^{b+1}+1}{4}E_b - \frac{3^{b+3}+1}{4}E_{b+2}\right) \pmod{2^{m+4}}. \end{aligned} \tag{3.1}$$

For $r \geq 3$ we see that $4^{r-3}/r$ is a 2-integer. Thus,

$$\begin{aligned} 3^{2^m}-1 &= (1-4)^{2^m}-1 = \sum_{r=1}^{2^m}\binom{2^m}{r}(-4)^r = \sum_{r=1}^{2^m}\frac{2^m}{r}\binom{2^m-1}{r-1}(-4)^r \\ &\equiv 2^m\cdot(-4) + 2^{m-1}(2^m-1)(-4)^2 \equiv -3\cdot 2^{m+2} \pmod{2^{m+6}}. \end{aligned}$$

Therefore,

$$3^{2^m k} \equiv (1-3\cdot 2^{m+2})^k \equiv 1-3k\cdot 2^{m+2} \pmod{2^{m+6}}$$

and so

$$\frac{3^{2^m k+b+1}+1}{4} \equiv \frac{(1-3k\cdot 2^{m+2})3^{b+1}+1}{4} = \frac{3^{b+1}+1}{4} - 3^{b+2}\cdot 2^m k \pmod{2^{m+4}}.$$

This together with (3.1) yields

$$\frac{3^{b+1}+1}{4}(E_{2^m k+b}-E_b)\equiv 2^m k\cdot 3^{b+2}E_{2^m k+b}-2^{m+2}k\cdot\frac{1}{8} \times\left(\frac{3^{b+1}+1}{4}E_b-\frac{3^{b+3}+1}{4}E_{b+2}\right) \mod 2^{m+4}. \tag{3.2}$$

It is clear that

$$\frac{3^{b+1}+1}{4}=\frac{3(1+8)^{b/2}+1}{4}\equiv\frac{3\left(1+(b/2)\cdot 8+\binom{b/2}{2}\cdot 8^2\right)+1}{4}$$
$$=1+3b+24\frac{b}{2}\left(\frac{b}{2}-1\right)\equiv\begin{cases}3b+1 \mod 32 & \text{if } b\equiv 0,2 \mod 8,\\ 3b-15 \mod 32 & \text{if } b\equiv 4,6 \mod 8,\end{cases}$$

and so

$$\frac{3^{b+3}+1}{4}\equiv\begin{cases}3(b+2)+1=3b+7 \mod 32 & \text{if } b\equiv 0,6 \mod 8,\\ 3(b+2)-15=3b-9 \mod 32 & \text{if } b\equiv 2,4 \mod 8.\end{cases}$$

From Corollary 3.5 or [5, Corollary 7.7] we know that

$$E_b\equiv\begin{cases}b+1 \mod 32 & \text{if } 4\mid b,\\ b-3 \mod 32 & \text{if } 4\mid b-2,\end{cases}$$

and so

$$E_{b+2}\equiv\begin{cases}b+3 \mod 32 & \text{if } 4\mid b-2,\\ b-1 \mod 32 & \text{if } 4\mid b.\end{cases}$$

Thus, modulo 32,

$$\frac{3^{b+1}+1}{4}E_b-\frac{3^{b+3}+1}{4}E_{b+2}$$
$$\equiv\begin{cases}(3b+1)(b+1)-(3b+7)(b-1)=8 & \text{if } b\equiv 0 \mod 8,\\ (3b+1)(b-3)-(3b-9)(b+3)=24-8b & \text{if } b\equiv 2 \mod 8,\\ (3b-15)(b+1)-(3b-9)(b-1)=-24 & \text{if } b\equiv 4 \mod 8,\\ (3b-15)(b-3)-(3b+7)(b+3)=-40b+24 & \text{if } b\equiv 6 \mod 8\end{cases} \tag{3.3}$$

and therefore

$$\frac{1}{8}\left(\frac{3^{b+1}+1}{4}E_b-\frac{3^{b+3}+1}{4}E_{b+2}\right)\equiv 1 \mod 4.$$

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Substituting this into (3.2), we obtain

$$\frac{3^{b+1}+1}{4}(E_{2^mk+b}-E_b)\equiv 2^mk\cdot 3^{b+2}E_{2^mk+b}-2^{m+2}k \mod 2^{m+4}.$$

Since $(3^{b+1}+1)/4$ is odd, we deduce that

$$E_{2^mk+b}-E_b\equiv \frac{2^mk}{(3^{b+1}+1)/4}(3^{b+2}E_{2^mk+b}-4) \mod 2^{m+4}. \tag{3.4}$$

From the previous argument, $(3^{b+1}+1)/4\equiv 3b+1 \bmod 16$ and

$$E_{2^mk+b}\equiv\begin{cases}2^mk+b+1 \mod 16 & \text{if } b\equiv 0 \mod 4,\\ 2^mk+b-3 \mod 16 & \text{if } b\equiv 2 \mod 4.\end{cases}$$

Thus,

$$\frac{1}{(3^{b+1}+1)/4}\equiv\frac{1}{3b+1}\equiv\begin{cases}1-3b\equiv 1+b \mod 16 & \text{if } b\equiv 0 \mod 4,\\ 13-3b\equiv -3(1+b) \mod 16 & \text{if } b\equiv 2 \mod 4.\end{cases}$$

Furthermore,

$$3^{b+2}=\begin{cases}9\cdot 81^{b/4}\equiv 9 \mod 16 & \text{if } b\equiv 0 \mod 4,\\ 81^{(b+2)/4}\equiv 1 \mod 16 & \text{if } b\equiv 2 \mod 4.\end{cases}$$

Hence, by the above and (3.4),

$$E_{2^mk+b}-E_b\equiv\begin{cases}(1+b)\cdot 2^mk(9(2^mk+b+1)-4) \mod 2^{m+4} & \text{if } 4\mid b,\\ -3(1+b)\cdot 2^mk((2^mk+b-3)-4) \mod 2^{m+4} & \text{if } 4\mid b-2.\end{cases}$$

For $m=3$ we have $2^{2m}=2^{m+3}$. For $m\ge 4$ we have $2^{m+4}\mid 2^{2m}$. Therefore

$$E_{2^mk+b}-E_b\equiv\begin{cases}(1+b)\cdot 2^mk(9b+5)+2^{m+3}[3/m]k \mod 2^{m+4} & \text{if } 4\mid b,\\ -3(1+b)\cdot 2^mk(b-7)+2^{m+3}[3/m]k \mod 2^{m+4} & \text{if } 4\mid b-2.\end{cases}$$

It is easily seen that

$$(1+b)(9b+5)=9b^2+14b+5\equiv 2b+5\equiv\begin{cases}5 \mod 16 & \text{if } 8\mid b,\\ -3 \mod 16 & \text{if } 8\mid b-4\end{cases}$$

and

$$\begin{aligned}-3(1+b)(b-7)&=-3(b-2)^2+6b+33\equiv -2b+1\\ &\equiv\begin{cases}5 \mod 16 & \text{if } 8\mid b-6,\\ -3 \mod 16 & \text{if } 8\mid b-2.\end{cases}\end{aligned}$$

Thus the result follows. □

COROLLARY 3.10. *Let $b\in\{0,2,4,\ldots\}$ and $k,m\in\mathbb{N}$ with $m\ge 3$. Then*

$$E_{2^mk+b}\equiv E_b+5\cdot 2^mk \mod 2^{m+3}.$$

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References

[1] M. A. Stern, 'Zur Theorie der Eulerschen Zahlen', *J. Reine Angew. Math.* **79** (1875), 67–98.

[2] Z. H. Sun, 'Congruences for Bernoulli numbers and Bernoulli polynomials', *Discrete Math.* **163** (1997), 153–163.

[3] Z. H. Sun, 'Congruences concerning Bernoulli numbers and Bernoulli polynomials', *Discrete Appl. Math.* **105** (2000), 193–223.

[4] Z. W. Sun, 'On Euler numbers modulo powers of two', *J. Number Theory* **115** (2005), 371–380.

[5] Z. H. Sun, 'Congruences involving Bernoulli polynomials', *Discrete Math.* **308** (2008), 71–112.

[6] S. S. Wagstaff Jr, 'Prime divisors of the Bernoulli and Euler numbers', in: *Number Theory for the Millennium,* Vol. III (Urbana, IL, 2000), (eds. M. A. Bennett *et al.*) (AK Peters, Natick, MA, 2002), pp. 357–374.

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