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Quasivarieties and varieties of ordered algebras: regularity and exactness†

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We characterise quasivarieties and varieties of ordered algebras categorically in terms of regularity, exactness and the existence of a suitable generator. The notions of regularity and exactness need to be understood in the sense of category theory enriched over posets.
We also prove that finitary varieties of ordered algebras are cocompletions of their theories under sifted colimits (again, in the enriched sense).

1. Introduction

Since the very beginning of the categorical approach to universal algebra, the intrinsic characterisation of varieties and quasivarieties of algebras has become an interesting question. First steps were taken already in John Isbell's paper (Isbell 1964), William Lawvere's seminal PhD thesis (Lawvere 1963) and Fred Linton's paper (Linton 1966). The compact way of characterising varieties and quasivarieties can be, in modern language, perhaps best stated as follows:

> A category $\mathscr{A}$ is equivalent to a (quasi)variety of algebras iff it is (regular) exact and it possesses a 'nice' generator.

For the excellent modern categorical treatment of (quasi)varieties of algebras in the sense of classical universal algebra, see the book by Adámek *et al.* (2011).

In the current paper, we will give a characterisation of categories of varieties and quasivarieties of *ordered* algebras in essentially the same spirit:

> A category $\mathscr{A}$, *enriched over posets*, is equivalent to a (quasi)variety of *ordered* algebras iff it is (regular) exact and it possesses a 'nice' generator.

Above, however, the notions of regularity and exactness need to be reformulated so that the notions suit the realm of categories enriched over posets.

There are at least two approaches to what an ordered algebra can be. Let us briefly comment on both:

The approach of Bloom and Wright (1983) A *signature* Σ specifies for each natural number n a set Σn of operation symbols. An *algebra* for a signature Σ consists of a poset

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X, together with a monotone map $[\![\sigma]\!] : X^n \longrightarrow X$, for each specified n-ary operation σ, where n is a *set*. A *homomorphism* is a monotone map, preserving the operations on the nose.

Such a concept is a direct generalisation of the classical notion of an algebra (Cohn 1981).

The approach of Kelly and Power (1993) A *signature* Σ specifies for each finite poset n a set Σn of operation symbols. An *algebra* for a signature Σ consists of a poset X, together with a monotone map $[\![\sigma]\!] : X^n \longrightarrow X$, for each specified n-ary operation σ. Here, X^n denotes the poset of all *monotone* maps from n to X. A *homomorphism* is a monotone map, preserving the operations on the nose.

This concept stems from the theory of enriched monads. It allows for operations that are defined only *partially*. As we will see later, such an approach is also quite natural and handy in practice.

We will choose the first concept as the object of our study. For technical reasons, we will also allow the collection Σn of all n-ary operations to be a poset. Then, for every algebra for Σ on a poset X, the inequality $[\![\sigma]\!] \leqslant [\![\tau]\!]$ is required to hold in the poset of monotone functions from X^n to X, whenever $\sigma \leqslant \tau$ holds in the poset Σn of all n-ary operations.

Varieties and quasivarieties in the first sense were studied by Stephen Bloom and Jesse Wright in Bloom (1976) and Bloom and Wright (1983). In Bloom (1976), a Birkhoff-style characterisation of classes of algebras is given follows:

1. Varieties are *defined* as classes of algebras satisfying *formal inequalities* of the form

$$t' \sqsubseteq t$$

where t' and t are Σ-terms. Varieties can be *characterised* as precisely the HSP-classes of Σ-algebras.

2. Quasivarieties are *defined* as classes of algebras that satisfy *formal implications* (or, *quasi-inequalities*) of the form

$$(\bigwedge_{i \in I} s_i' \sqsubseteq s_i) \Rightarrow t' \sqsubseteq t$$

where I is a set, s_i', s_i, t' and t are Σ-terms. Quasivarieties can be *characterised* as precisely the SP-classes of Σ-algebras.

One has to be precise, however, in saying what the closure operators H and S mean. As it turns out, when choosing monotone surjections as the notion of a homomorphic image, then the proper concept of a subalgebra is that of a monotone homomorphism that reflects the order. This means that a subalgebra inherits not only the algebraic structure but also the order structure.

Example 1.1 (Varieties).

1. Since a signature in the sense of Bloom and Wright specifies the same data as a signature in the sense of ordinary universal algebra, it is the case that any ordinary variety is contained in the variety of ordered algebras for the same signature and equations, the ordinary algebras appearing as the algebras with a discrete order. This

gives rise to a large class of examples which includes ordered monoids and ordered semigroups, which play an important role in automata theory, see Pin (1997).
2. In some important examples such as Boolean algebras, Heyting algebras or groups, the discrete order is the only order that makes all operations monotone. In this case, the varieties are just the ordinary varieties.
3. Another important source of examples are ordinary varieties which have a semi-lattice reduct. Then the order is equationally definable via $x \leqslant y \Leftrightarrow x \vee y = y$. These examples are varieties in the ordered sense if equipped with the above order, but cannot be expected to be varieties in the ordered sense if they are equipped with the discrete order, see the comment following Example 3.21.
4. Ordered algebras arise naturally as solutions of domain equations. In order to build semantic domains which contain both infinite elements (to give semantics to loops and recursion) and their finite approximants, one works with ordered algebras, the partial order capturing the order of approximation, see e.g. Scott (1971). For example, infinite lists over a set A and their finite approximants arise as a certain algebra for the functor $F : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ given by

$$FX = \{\perp\} + A \times X = \{(a, x) \mid a \in A, x \in X\}/(\forall z.\perp \leqslant z)$$

F-algebras form a variety with the signature given by one constant $\perp$ and unary operation symbols a for all $a \in A$ and with one inequality given by $\forall z.\perp \leqslant z$. The initial algebra contains all finite lists over A and is ordered by the prefix relation. Its completion by ω-chains or directed joins (Goguen *et al.* 1977) also contains the infinite lists over A. This algebra can also be elegantly described as the final F-coalgebra.

Example 1.2 (Quasivarieties).

1. Sets and mappings form a quasivariety of ordered algebras. More precisely:
   a. Let Σ be a signature with no specified operation. Hence, Σ-algebras are exactly the posets and Σ-homomorphisms are the monotone maps.
   b. Let the objects be Σ-algebras, subject to the implication

$$x \sqsubseteq y \Rightarrow y \sqsubseteq x.$$

Clearly, any object can be identified with a set and Σ-homomorphisms can be identified with mappings.
It is easy to see that Set is an SP-class in the category of all Σ-algebras. But it is not an HSP-class: consider the identity-on-objects monotone mapping $e : 2 \longrightarrow \mathbb{2}$, where 2 is the discrete poset on two elements and $\mathbb{2}$ is the two-element chain. Then 2 is an object of $\mathscr{A}$, while $\mathbb{2}$ is not.
2. More generally, since discreteness is definable by quasi-inequalities, any ordinary quasivariety can be considered as a quasivariety of discrete algebras.
3. Every ordinary quasivariety gives rise to a quasivariety of ordered algebras for the same signature and quasi-equations. For example, the quasivariety of cancellative ordered monoids is a quasivariety which is not monadic in the ordered sense for the same reason that cancellative monoids are not monadic in the ordinary sense.

4. The Kleene algebras of Kozen (1994), which axiomatize the algebra of regular languages, form a quasivariety of algebras in the order induced by the underlying semiring.
5. Continuing from Example 1.1(4), recursive functions on data-types are conveniently defined using quasi-equations. For example, a function *member*$(a, -)$ which takes as an argument a list and returns a truth-value may be defined on finite lists as follows. It would be natural to formulate this in a many-sorted setting, see Remark 5.11, but for the purposes of this example just assume that we added a constant *true* to our signature. Then

$$\begin{aligned} & \mathit{member}(a, al) = \mathit{true} \\ \mathit{member}(a, l) = \mathit{true} \ \Rightarrow\ & \mathit{member}(a, bl) = \mathit{true} \end{aligned}$$

is an axiomatic description of a function *member*.

Example 1.3 (Monadic categories).

1. To continue from Example 1.2(1), the obvious discrete-poset functor $U : \mathsf{Set} \longrightarrow \mathsf{Pos}$ is easily seen to be monadic. Hence, Set appears as a 'variety' in the world where arities can be posets. More precisely: consider the signature Γ, where $\Gamma\mathbb{2} = \mathbb{2}$ and $\Gamma n = \varnothing$ otherwise. Then the set of equations

$$\sigma_0(x, y) = y, \quad \sigma_1(x, y) = x,$$

defines Set over Pos equationally, where $\sigma_0 \leqslant \sigma_1$ are the only elements of $\Gamma\mathbb{2}$. See Kelly and Power (1993) for more details on presenting monads by operations and equations.
2. More generally, any ordinary variety gives rise to a monadic category of discretely ordered algebras. Indeed, given a monadic functor $\mathcal{K} \longrightarrow \mathsf{Set}$ composition with the discrete-poset functor $\mathsf{Set} \longrightarrow \mathsf{Pos}$ is monadic, as can easily be checked using Beck's theorem.
3. An example of a monadic category that is not a quasivariety is given in Example 5.12.
4. Examples of quasivarieties that are not monadic are cancellative monoids or Kleene algebras.

Remark 1.4 ((Quasi)varieties vs. monadic categories).

1. Despite of the focus on (quasi)varieties, monadic categories still play an important role both in Bloom and Wright (1983) and in this paper. Indeed, an important ingredient of the main results is the following relationship: Every quasivariety is the full reflective subcategory of a monadic category for a surjection-preserving monad and every monadic category for a surjection-preserving monad is a quasivariety.
2. From the point of view of monads, one may ask why the restriction to surjection-preserving monads is of special interest. One of the important good properties of universal algebra that one loses beyond the surjection-preserving situation is that quotients are surjections. Indeed, for general monads on Pos it may happen that the quotient $A \longrightarrow A'$ of an algebra A by additional inequalities is not onto. The reason is that in the presence of ordered arities, adding new inequations may lead to the generation of new terms. Hyland and Power (2006) give further reasons why the restriction to discrete arities can be of interest.

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3. To summarize, the ordered universal algebra of Bloom and Wright (1983) studied in this paper is concerned with a universal algebra where taking a quotient means to add further inequations without generating new elements.
4. In the finitary case, a satisfactory reconciliation of varieties and monadic categories is obtained in Theorem 6.9: The finitary varieties are the monadic categories for sifted-colimits-preserving monads.
5. Let us also emphasize that both in the approach of Bloom and Wright (1983) and in the approach of Kelly and Power (1993) operations are monotone. An investigation into a categorical universal algebra where operations such as negation, implication, or inverse are antitone is outside the scope of this paper.

The system (monotone surjective maps, monotone maps reflecting orders) is a factorisation system in the category Pos of posets and monotone maps. One can therefore ask whether this system can play the role of the (regular epi, mono) factorisation system on the category of sets that is so vital in giving intrinsic categorical characterisations of varieties and quasivarieties in classical universal algebra. We prove that this is the case, if we pass from the world of categories to the world of categories *enriched* in posets. Namely:

1. We give the definition of regularity and exactness of a category enriched in posets. We show that Pos is an exact category.
2. We give intrinsic characterisations of both varieties and quasivarieties of ordered algebras, see Theorems 5.9 and 5.13 below. Our main results then have the same phrasing as in the classical case, the only difference is that all the notions have their meaning in category theory enriched in posets.

1.1. *Related work*

The notion of regularity and exactness for 2-categories goes back to Street (1982), but we were also much inspired by its polished version of Mike Shulman (http://ncatlab.org/nlab/show/2-congruence) and the recent PhD thesis of Bourke (2010). Bourke studies exactness for a different factorisation system, though. After our submission, we learned of Bourke and Garner (2014), where general notions of regularity and exactness with respect to a factorisation system are studied in the realm of enriched category theory. Varieties and quasivarieties from the current text were named P-varieties and P-quasivarieties by Stephen Bloom and Jesse Wright in Bloom and Wright (1983). The authors did not use the standard terminology and they only worked with kernel pairs (see Remark 3.9) and not with congruences (Definition 3.8) and hence they missed the notion of exactness. However, they give an 'almost intrinsic' characterisation of varieties and quasivarieties that we found extremely useful.

1.2. *Organisation of the text*

The necessary notions of enriched category theory are recalled in Section 2. Regularity and exactness are defined in Section 3. Section 4 contains the technicalities that we need in order to prove our main characterisation results in Section 5. We prove in Section 6, that finitary varieties of ordered algebras can be characterised as algebras for a special class of monads — the strongly finitary ones. In Section 7, we indicate directions for future work.

2. Preliminaries

We briefly recall the basic notions of enriched category that we will use later on. For more details, see Max Kelly's book (Kelly 2005).

We will work with categories enriched in the cartesian closed category $(\mathsf{Pos}, \times, \mathbb{1})$ of posets and monotone maps. We will omit the prefix Pos- when speaking of Pos-categories, Pos-functors, etc. Thus, in what follows:

1. A *category* $\mathscr{X}$ is given by objects X, Y, ... such that every hom-object $\mathscr{X}(X, Y)$ is a poset. The partial order on $\mathscr{X}(X, Y)$ is denoted by $\leqslant$. We require the composition to preserve the order in both arguments: $(g' \cdot f') \leqslant (g \cdot f)$ holds, whenever $g' \leqslant g$ and $f' \leqslant f$.
2. A *functor* $F : \mathscr{A} \longrightarrow \mathscr{B}$ is given by the functorial object-assignment that is locally monotone, i.e. $Ff \leqslant Fg$ holds, whenever $f \leqslant g$.

When we want to speak of non-enriched categories, functors, etc., we will call them *ordinary*.

Example 2.1. The category Set is the category of sets and functions with discretely ordered hom-sets. The category Pos is the category of posets and monotone maps with the order on hom-sets $\mathsf{Pos}(X, Y)$ induced by Y, that is, $f \leqslant g$ if $f(x) \leqslant g(x)$ for all $x \in X$. As ordinary categories Set and Pos do not have an order on their hom-sets. This distinction is relevant: there is the 'discrete' functor $D : \mathsf{Set} \longrightarrow \mathsf{Pos}$, but forgetting the order is only an ordinary functor $\mathsf{Pos} \longrightarrow \mathsf{Set}$. On the other hand, there is a 'connected component' functor $\mathsf{Pos} \longrightarrow \mathsf{Set}$, which is left-adjoint to the discrete functor D.

Remark 2.2. Categories and functors are called *P-categories* and *P-functors* in Bloom and Wright (1983).

In diagrams, we will denote, for parallel morphisms f, g, the fact $f \leqslant g$ by an arrow between morphisms and we will speak of a *2-cell*:

$$\begin{tikzcd} X \arrow[r, bend left, "g"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\uparrow"] & Y \end{tikzcd}$$

This notation complies with the fact that categories enriched in posets are (rather special) 2-categories.

The category of functors from $\mathscr{A}$ to $\mathscr{B}$ and natural transformations between them is denoted by $[\mathscr{A}, \mathscr{B}]$. The opposite category $\mathscr{X}^{op}$ of $\mathscr{X}$ has just the sense of morphisms reversed, the order on hom-posets remains unchanged.

The proper concept of a limit and a colimit in enriched category theory is that of a weighted (co)limit. In our setting, the main reason for this is that pullbacks and

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coequalizers need to be adapted to inequalities. In particular, comma objects will replace pullbacks in the construction of kernels and coinserters will replace coequalizers in the construction of quotients.

More in detail, for every *diagram* $D : \mathscr{D} \longrightarrow \mathscr{X}$, $\mathscr{D}$ small, we define its *tilde-conjugate*

$$\widetilde{D} : \mathscr{X} \longrightarrow [\mathscr{D}^{op}, \mathsf{Pos}], \quad X \mapsto \mathscr{X}(D-, X)$$

and its *hat-conjugate*

$$\widehat{D} : \mathscr{X} \longrightarrow [\mathscr{D}, \mathsf{Pos}]^{op}, \quad X \mapsto \mathscr{X}(X, D-).$$

Then, a *colimit* of D weighted by $W : \mathscr{D}^{op} \longrightarrow \mathsf{Pos}$ is an object $W * D$, together with an isomorphism

$$\mathscr{X}(W * D, X) \cong [\mathscr{D}^{op}, \mathsf{Pos}](W, \widetilde{D}X)$$

of posets, natural in X. A *limit* of D weighted by $W : \mathscr{D} \longrightarrow \mathsf{Pos}$ is an object $\{W, D\}$, together with an isomorphism

$$\mathscr{X}(X, \{W, D\}) \cong [\mathscr{D}, \mathsf{Pos}]^{op}(\widehat{D}X, W)$$

of posets, natural in X.

Hence, for a category $\mathscr{X}$ admitting all colimits of the diagram $D : \mathscr{D} \longrightarrow \mathscr{X}$, the assignment $X \mapsto X * D$ is the value of a left adjoint to $\widetilde{D} : \mathscr{X} \longrightarrow [\mathscr{D}^{op}, \mathsf{Pos}]$. A special instance is the case of a one-morphism category $\mathscr{D}$: the diagram $D : \mathscr{D} \longrightarrow \mathscr{X}$ can be identified with an object D of $\mathscr{X}$, the functor $\widetilde{D}$ is the representable functor $\mathscr{X}(D, -) : \mathscr{X} \longrightarrow \mathsf{Pos}$ and its left adjoint assigns the *tensor* $X \bullet D$ of the object D and the poset X.

Analogously, the assignment $X \mapsto \{X, D\}$ is a right adjoint to $\widehat{D} : \mathscr{X} \longrightarrow [\mathscr{D}, \mathsf{Pos}]^{op}$ in case $\mathscr{X}$ admits all limits of $W : \mathscr{D} \longrightarrow \mathscr{X}$.

Recall from Kelly (1982) that a (co)limit is finite, if it is weighted by a finite weight. The latter is a functor $W : \mathscr{D} \longrightarrow \mathsf{Pos}$ such that $\mathscr{D}$ has finitely many objects, every $\mathscr{D}(d', d)$ is a finite poset, and every Wd is a finite poset.

We will, besides other finite (co)limits, use *comma objects* and *coinserters*.

1. A comma object is a weighted limit. The weight $W : \mathscr{D} \longrightarrow \mathsf{Pos}$ for comma objects is the functor

$$\begin{tikzcd} & b \arrow[d, "g"] \\ a \arrow[r, "f"'] & c \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} & \mathbb{1} \arrow[d, "1"] \\ \mathbb{1} \arrow[r, "0"'] & \mathbb{2} \end{tikzcd}$$

In elementary terms, a comma object in $\mathscr{X}$ of a diagram

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

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is a 'lax commutative square' of the form

$$\begin{tikzcd} f/g \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\nearrow"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

(i.e. the inequality $f \cdot p_0 \leqslant g \cdot p_1$ holds) that satisfies the following universal property:

a. Given any 'lax commutative square'

$$\begin{tikzcd} Z \arrow[r, "h_1"] \arrow[d, "h_0"'] \arrow[dr, phantom, "\nearrow"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

there is a unique $h : Z \longrightarrow f/g$ such that $p_0 \cdot h = h_0$ and $p_1 \cdot h = h_1$.

b. For any parallel pair $k', k : Z \longrightarrow f/g$ of morphisms such that $p_0 \cdot k' \leqslant p_0 \cdot k$ and $p_1 \cdot k' \leqslant p_1 \cdot k$, the inequality $k' \leqslant k$ holds.

2. A coinserter is a weighted colimit. The weight $W : \mathscr{D}^{op} \longrightarrow \mathsf{Pos}$ for coinserters has $\mathscr{D}$ consisting of a parallel pair of morphisms that is sent to the parallel pair

$$\mathbb{1} \underset{0}{\overset{1}{\rightrightarrows}} \mathbb{2}$$

in Pos. In elementary terms, a coinserter in $\mathscr{X}$ of a parallel pair

$$X_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} X_0$$

consists of a morphism $c : X_0 \longrightarrow C$ such that $c \cdot d_0 \leqslant c \cdot d_1$ holds and such that it satisfies the following couniversal property:

a. For any $h : X_0 \longrightarrow D$ such that $h \cdot d_0 \leqslant h \cdot d_1$ there is a unique $h^\sharp : C \longrightarrow D$ such that $h^\sharp \cdot c = h$.

b. For any pair $k', k : C \longrightarrow D$ that satisfies $k' \cdot c \leqslant k \cdot c$, the inequality $k' \leqslant k$ holds.

Thus the (co)universal property of a (co)limit has two aspects: the 1-dimensional aspect (concerning 1-cells) and the 2-dimensional aspect (concerning the order between 1-cells). This will be always the case for weighted (co)limits that we encounter and it is caused by the fact that we enrich over posets. As such, our (co)limits will be rather special 2-(co)limits. The enrichment in posets will usually simplify substantially the 2-dimensional aspect of 2-(co)limits. See Kelly (1989) for more details.

Example 2.3 (Explicit computation of comma objects in Pos). Suppose that a diagram

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

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in Pos is given. The 'lax commutative square'

$$\begin{tikzcd} f/g \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, "\nearrow"] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

is a comma object of the above diagram, where by f/g we denote the poset of all pairs (a,b) of elements of A and B such that $fa \leqslant gb$ holds in C. The pairs (a,b) are ordered pointwise, using the orders of A and B. The monotone maps $p_0 : f/g \longrightarrow A$ and $p_1 : f/g \longrightarrow B$ are the projections.

Example 2.4 (Explicit computation of coinserters in Pos). Suppose that

$$X_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} X_0$$

is a pair of morphisms in Pos. The coinserter $c : X_0 \longrightarrow C$ of d_0, d_1 is obtained by adding for all $f \in X_1$ an inequality $d_0(f) \leqslant d_1(f)$ to X_0 and can be described formally as follows:

1. Define a binary relation R on the set $\mathrm{ob}(X_0)$ of objects of X_0 as follows:

   $x'\ R\ x$ iff there is a finite sequence $f_0, \ldots, f_{n-1}$ of objects in X_1 such that the inequalities

   $$x' \leqslant d_0(f_0), \quad d_1(f_0) \leqslant d_0(f_1), \quad d_1(f_1) \leqslant d_0(f_2), \quad \ldots, \quad d_1(f_{n-1}) \leqslant x$$

   hold in X_0.

   It is easy to see that R is reflexive and transitive. Put $E = R \cap R^{op}$ to obtain an equivalence relation on the set $\mathrm{ob}(X_0)$.
2. The poset C has as $\mathrm{ob}(C)$ the quotient set $\mathrm{ob}(X_0)/E$, we put $[x'] \leqslant [x]$ in C to hold iff $x'\ R\ x$ holds. The monotone mapping $c : X_0 \longrightarrow C$ is the canonical map sending x to $[x]$.

It is now routine to verify that we have defined a coinserter.

Remark 2.5. Comma objects f/f are called *P-kernels* and coinserters are called *P-coequalizers* in Bloom and Wright (1983).

3. Regularity and exactness

Regularity and exactness in ordinary category theory (Barr *et al.* 1971) is defined relative to a factorisation system. In this section, we will introduce the factorisation system

(surjective on objects, representably fully faithful)

on the class of morphisms of a general category $\mathscr{X}$. When $\mathscr{X} = $ Pos, the above system coincides with the factorisation system (monotone surjective maps, monotone maps reflecting orders).

We introduce the factorisation system by starting with its 'mono' part. The 'strong epi' part of the factorisation system is then derived by the orthogonal property that is appropriate for the enrichment in posets. We then show that, in cases of interest, the 'strong epi' part of the factorisation system is given by a suitable generalisation of a coequalizer. This is the gist of the second part of this section: we introduce congruences and their quotients and the corresponding notions of regularity and exactness.

3.1. *The factorisation system*

Definition 3.1. We say that $m : X \longrightarrow Y$ in $\mathscr{X}$ is *representably fully faithful* (or, that it is an ff-morphism), provided that the monotone map $\mathscr{X}(Z,m) : \mathscr{X}(Z,X) \longrightarrow \mathscr{X}(Z,Y)$ reflects orders (i.e. if it is fully faithful as a functor in Pos), for every Z.

A morphism $e : A \longrightarrow B$ is *surjective on objects* (or, that it is an so-morphism), provided that the square

$$\begin{tikzcd} \mathscr{X}(B,X) \arrow[r, "\mathscr{X}(e,X)"] \arrow[d, "\mathscr{X}(B,m)"'] & \mathscr{X}(A,X) \arrow[d, "\mathscr{X}(A,m)"] \\ \mathscr{X}(B,Y) \arrow[r, "\mathscr{X}(e,Y)"'] & \mathscr{X}(A,Y) \end{tikzcd} \tag{3.1}$$

is a pullback in Pos, for every ff-morphism $m : X \longrightarrow Y$.

We say that $\mathscr{X}$ *has* $(\mathsf{so},\mathsf{ff})$-*factorisations* if every f can be factored as an so-morphism followed by an ff-morphism.

Example 3.2. An ff-morphism is necessarily mono. In Pos, so-morphisms are exactly the monotone surjections, ff-morphisms are order-reflecting monotone maps. Clearly, Pos has $(\mathsf{so},\mathsf{ff})$-factorisations.

The description extends to 'presheaf' categories $[\mathscr{S}^{op},\mathsf{Pos}]$, where $\mathscr{S}$ is small, in the usual 'pointwise' way.

Remark 3.3. The ff-morphisms are called *P-monics* in Bloom and Wright (1983), and *chronic* in Street (1982). We choose the acronym ff to remind us of (representably) *fully faithful* Street and Walters (1978). The so-morphisms are called *surjections* in Bloom and Wright (1983) and *acute* in Street (1982). Our justification to replace familiar terminology from posets such as monotone order-reflecting map by categorical terminology such as ff-morphism is the following. The main idea of our approach here is to specialise methods that work for categories enriched over categories to categories enriched over posets. Not surprisingly, by going from categories to posets, notions that are different for categories over categories collapse for categories over posets. Nevertheless it seems important to us to use a terminology that remains valid if going to the richer setting of categories over categories.

Remark 3.4. We defined the factorisation system in the manner that is common in enriched category theory. More precisely, we chose the 'monos' and defined the 'epis' via orthogonality expressed by a pullback in the base category of posets. That the diagram (3.1) is a pullback on the level of sets states the usual 'diagonal fill-in' property. Hence, classes of so-morphisms and ff-morphisms are mutually *orthogonal*. This means that in every commutative square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "v"] \arrow[dl, dotted, "d"'] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

with e an so-morphism and m an ff-morphism, there is a unique diagonal d as indicated, making both triangles commutative.

That the diagram (3.1) is in fact a pullback on the level of posets describes a finer, *2-dimensional aspect* of orthogonality.

Namely, for two pairs $u_1 \leqslant u_2 : A \longrightarrow X$, $v_1 \leqslant v_2 : B \longrightarrow Y$ such that both squares

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u_1"'] & B \arrow[d, "v_1"] \\ X \arrow[r, "m"'] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u_2"'] & B \arrow[d, "v_2"] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

commute, we have an inequality $d_1 \leqslant d_2$ for the respective diagonals. In fact, the 2-dimensional aspect can be omitted here, since it follows from the fact that m is ff.

3.2. *Congruences and their quotients*

We will define congruences and their quotients.† Since the general poset-enriched concept of a congruence is rather technical, we start with the following intuition for equivalence relations on sets:

> An equivalence relation E on a set X is a 'recipe' how to glue elements of X together. That is: E imposes *new equations* on the set X, besides those already valid.

Remark 3.5 (Category object, equivalence relation). In a category with finite limits an *equivalence relation* (Barr *et al.* 1971; Duskin 1969) is a diagram

$$\begin{tikzcd} A_2 \arrow[r, shift left=2, "d_2^2"] \arrow[r, "d_1^2" description] \arrow[r, shift right=2, "d_0^2"'] & A_1 \arrow[loop above, "s"] \arrow[r, shift left=2, "d_1^1"] \arrow[r, shift right=2, "d_0^1"'] & A_0 \arrow[l, "i_0^0" description] \end{tikzcd}$$

where

1. the square $\begin{tikzcd} A_2 \arrow[r, "d_2^2"] \arrow[d, "d_0^2"] & A_1 \arrow[d, "d_0^1"] \\ A_1 \arrow[r, "d_1^1"] & A_0 \end{tikzcd}$ is a pullback
2. $d_1^1 \circ d_1^2 = d_1^1 \circ d_2^2$, $d_0^1 \circ d_1^2 = d_0^1 \circ d_0^2$,

† The standard terminology of 2-category theory for quotients is *codescent*, see Lack (2002) or Bourke (2010). We prefer to use the term quotient to comply with the intuitions of classical universal algebra.

3. $d_1^1 \circ i_0^0 = d_0^1 \circ i_0^0 = id$,
4. $\langle d_0^1, d_1^1 \rangle : A_1 \longrightarrow A_0 \times A_0$ is mono,
5. $d_1^1 \circ s = d_0^1$, $d_0^1 \circ s = d_1^1$.

Without s and under conditions (1)–(3) one speaks of a *category object*. (4) expresses that A_1 together with projections d_0^1, d_1^1 is a relation on A_0, (1–2) say that the relation is transitive, (3) says that it is reflexive, and s and (5) are required to express symmetry. The quotient of A_0 by the equivalence relation can be computed as the coequalizer of d_0^1, d_1^1. □

A congruence E on a poset X should impose *new inequalities* besides those already valid. Moreover, E should be a poset again. Hence, an 'element' of a congruence E should be a formal 'broken' arrow $x' \nrightarrow x$ that specifies the formal inequality x' *is smaller than* x. The formal arrows should interact nicely with the actual arrows (representing already valid inequalities in X), i.e. both $x'' \longrightarrow x' \nrightarrow x$ and $x' \nrightarrow x \longrightarrow x''$ should have an unambiguous meaning (and both should compose to a 'broken' arrow). Furthermore, 'broken' arrows should compose (imposing inequalities is reflexive and transitive).

The above can be stated more formally: a congruence is a category object, whose domain-codomain span is a two-sided discrete fibration of a certain kind. Before giving the precise definition (Definition 3.8 below), let us see an example of a congruence in Pos:

Example 3.6 (Kernel congruences in posets). Every morphism $f : A_0 \longrightarrow B$ in Pos gives rise to a *kernel congruence* $\ker(f)$ on A_0 as follows:

1. Form a comma object

$$\begin{tikzcd} A_1 \arrow[r, "d_1^1"] \arrow[d, "d_0^1"'] \arrow[dr, phantom, "\nearrow"] & A_0 \arrow[d, "f"] \\ A_0 \arrow[r, "f"'] & B \end{tikzcd}$$

That is: objects of A_1 are pairs (a, b) such that $fa \leqslant fb$ holds in B. The pair (a, b) should be thought of as a new inequality that we want to impose. We denote such a formal inequality by $a \nrightarrow b$.

The pairs (a, b) in A_1 inherit the order from the product $A_0 \times A_0$. In other words: the map $\langle d_0^1, d_1^1 \rangle : A_1 \longrightarrow A_0 \times A_0$ is an ff-morphism.

It will be useful to denote the inequality $(a, b) \leqslant (a', b')$ in A_1 by a formal square

$$\begin{tikzcd} a \arrow[r, "\shortmid" marking] \arrow[d] & b \arrow[d] \\ a' \arrow[r, "\shortmid" marking] & b' \end{tikzcd}$$

Observe that there is an associative and unital way of *vertical composition* of formal squares by pasting one on top of another.

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It is well-known (see, e.g. Street (1974)) that the span (d_0^1, A_1, d_1^1) is a (two-sided) *discrete fibration*. This means that for every pair

$$
\begin{tikzcd}
a \arrow[d] & \\
a' \arrow[r, "\shortmid" marking] & b
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
a \arrow[r, "\shortmid" marking] & b \arrow[d] \\
 & b'
\end{tikzcd}
$$

of 'niches' there are 'unique fill-ins' of the form

$$
\begin{tikzcd}
a \arrow[r, "\shortmid" marking] \arrow[d] & b \arrow[d, equal] \\
a' \arrow[r, "\shortmid" marking] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\shortmid" marking] \arrow[d, equal] & b \arrow[d] \\
a \arrow[r, "\shortmid" marking] & b'
\end{tikzcd}
$$

and that every formal square

$$
\begin{tikzcd}
a \arrow[r, "\shortmid" marking] \arrow[d] & b \arrow[d] \\
a' \arrow[r, "\shortmid" marking] & b'
\end{tikzcd}
$$

can be written uniquely as a vertical composite

$$
\begin{tikzcd}
a \arrow[r, "\shortmid" marking] \arrow[d, equal] & b \arrow[d] \\
a \arrow[r, "\shortmid" marking] \arrow[d] & b' \arrow[d, equal] \\
a' \arrow[r, "\shortmid" marking] & b'
\end{tikzcd}
$$

of such fillings.

2. Besides pasting the formal squares vertically, we show how to paste them horizontally as in

$$
\begin{tikzcd}
a \arrow[r, "\shortmid" marking] \arrow[d] & b \arrow[d] \\
a' \arrow[r, "\shortmid" marking] & b'
\end{tikzcd}
\quad
\begin{tikzcd}
b \arrow[r, "\shortmid" marking] \arrow[d] & c \arrow[d] \\
b' \arrow[r, "\shortmid" marking] & c'
\end{tikzcd}
\quad \mapsto \quad
\begin{tikzcd}
a \arrow[r, "\shortmid" marking] \arrow[d] & c \arrow[d] \\
a' \arrow[r, "\shortmid" marking] & c'
\end{tikzcd}
$$

To allow for the *horizontal composition* of the squares, form a pullback

$$
\begin{tikzcd}
A_2 \arrow[r, "d_2^2"] \arrow[d, "d_0^2"'] & A_1 \arrow[d, "d_0^1"] \\
A_1 \arrow[r, "d_1^1"'] & A_0
\end{tikzcd}
$$

It is straightforward to see that the elements of A_2 are triples (a'', a', a) satisfying $fa'' \leqslant fa' \leqslant fa$. The triples are ordered pointwise. Every such triple (a'', a', a) can be drawn as a 'composable pair' $a'' \mathrel{\relbar\!\!\!\mapstochar\!\!\rightarrow} a' \mathrel{\relbar\!\!\!\mapstochar\!\!\rightarrow} a$. of 'broken' arrows. We now define two monotone maps

$$
d_1^2 : A_2 \longrightarrow A_1, \quad d_0^0 : A_0 \longrightarrow A_1,
$$

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with the intention that d_1^2 (the composition map) sends $a'' \nrightarrow a' \nrightarrow a$ to $a'' \nrightarrow a$ and $d_0^0 : A_0 \longrightarrow A_1$ (the identity map) sends each a in A_0 to the 'identity broken arrow' $a \nrightarrow a$.

One can use the universal property of the comma square to define $d_1^2 : A_2 \longrightarrow A_1$ as the unique map such that the equality

$$
\begin{tikzcd}
A_2 \arrow[dr, "d_1^2"] & & \\
& A_1 \arrow[r, "d_1^1"] \arrow[d, "d_0^1"'] \arrow[dr, phantom, "\nearrow"] & A_0 \arrow[d, "f"] \\
& A_0 \arrow[r, "f"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A_2 \arrow[r, "d_2^2"] \arrow[d, "d_0^2"'] & A_1 \arrow[r, "d_1^1"] \arrow[d, "d_0^1"'] \arrow[dr, phantom, "\nearrow"] & A_0 \arrow[d, "f"] \\
A_1 \arrow[r, "d_1^1"] \arrow[d, "d_0^1"'] \arrow[dr, phantom, "\nearrow"] & A_0 \arrow[r, "f"'] \arrow[d, "f"] & B \arrow[d, equal] \\
A_0 \arrow[r, "f"'] & B \arrow[r, equal] & B
\end{tikzcd}
$$

holds. It is clear that d_1^2 sends (a'', a', a) to (a'', a).

Analogously, one can define $i_0^0 : A_0 \longrightarrow A_1$ as the unique map such that the equality

$$
\begin{tikzcd}
A_0 \arrow[dr, "i_0^0"] & & \\
& A_1 \arrow[r, "d_1^1"] \arrow[d, "d_0^1"'] \arrow[dr, phantom, "\nearrow"] & A_0 \arrow[d, "f"] \\
& A_0 \arrow[r, "f"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A_0 \arrow[dr, "f"] & \\
& B
\end{tikzcd}
$$

holds. Explicitly: i_0^0 sends a to the pair (a, a).

To summarise: the above constructions yield a category object

$$
\ker(f) \equiv
\begin{tikzcd}[column sep=large]
A_2 \arrow[r, shift left=3, "d_2^2"] \arrow[r, "d_1^2" description] \arrow[r, shift right=3, "d_0^2"'] & A_1 \arrow[r, shift left=3, "d_1^1"] \arrow[r, leftarrow, "i_0^0" description] \arrow[r, shift right=3, "d_0^1"'] & A_0
\end{tikzcd}
$$

in Pos such that $\langle d_0^1, d_1^1 \rangle$ is an ff-morphism and the span (d_0^1, A_1, d_1^1) is a two-sided discrete fibration.

Remark 3.7. Clearly, the steps of the above construction of $\ker(f)$ can be performed in *any* category $\mathscr{X}$ admitting finite limits. In fact, the resulting category object will have the two additional properties as well, since:

1. A span (d_0^1, A_1, d_1^1) in a general category $\mathscr{X}$ is *defined* to be a two-sided discrete fibration if it is representably so. This means that the span $(\mathscr{X}(X, d_0^1), \mathscr{X}(X, A_1), \mathscr{X}(X, d_1^1))$ of monotone maps is a two-sided discrete fibration in Pos, for every X.
2. The morphism $\langle d_0^1, d_1^1 \rangle : A_1 \longrightarrow A_0 \times A_0$ is easily proved to be an ff-morphism in a general category $\mathscr{X}$ iff the morphism $\langle \mathscr{X}(X, d_0^1), \mathscr{X}(X, d_1^1) \rangle : \mathscr{X}(X, A_1) \longrightarrow \mathscr{X}(X, A_0) \times \mathscr{X}(X, A_0)$ is an ff-morphism in Pos, for every X.

The above considerations lead us to the following definition:

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Definition 3.8 (Street 1982; http://ncatlab.org/nlab/show/2-congruence). Suppose A_0 is an object of $\mathscr{X}$. We say that a category object

$$\sim \;\equiv\; A_2 \overset{d^2_2}{\underset{d^2_0}{\xrightarrow{\quad d^2_1 \quad}}} A_1 \overset{d^1_1}{\underset{d^1_0}{\xleftarrow{\quad i^0_0 \quad}}} A_0$$

in $\mathscr{X}$, where the span (d^1_0, A_1, d^1_1) is a (two-sided) discrete fibration and $\langle d^1_0, d^1_1\rangle : A_1 \longrightarrow A_0 \times A_0$ is an ff-morphism, is a *congruence* on A_0.

Remark 3.9. For a congruence $\sim$ as above, think of A_0 as the object of objects, A_1 as the object of morphisms, $i^0_0 : A_0 \longrightarrow A_1$ picks up the identity morphisms, $d^1_0 : A_1 \longrightarrow A_0$ is the domain map, $d^1_1 : A_1 \longrightarrow A_0$ is the codomain map, A_2 is the object of 'composable pairs of morphisms' (since A_2 is the vertex of a pullback of d^1_0 and d^1_1), in a composable pair, $d^2_0 : A_2 \longrightarrow A_1$ picks the 'morphism on the left,' $d^2_2 : A_2 \longrightarrow A_1$ picks the 'morphism on the right,' and $d^2_1 : A_2 \longrightarrow A_1$ is the composition.

The same notion, in the category of posets, is used in Pin (1997) in a non-categorical setting (Pin speaks of stable quasi-orders and also calls them congruences).

In Bloom and Wright (1983) a P-congruence is simply a pair (d^1_0, d^1_1) of morphisms that arises as a P-kernel (Remark 2.5) of some morphism. We need the more complicated notion of congruence because we want to show that varieties are those quasivarieties where all congruences arise as kernels.

To treat congruences and their quotients (the coinserters of the (d^1_0, d^1_1), see Remark 3.14) conceptually, let us introduce the following notation:

Notation 3.10 (Bourke 2010). Let $\mathbb{1}$, $\mathbb{2}$, $\mathbb{3}$ denote the chains on one, two, three elements, respectively. We denote by Δ^-_2 the simplicial category truncated at stage two and with the morphisms between stage three and stage two omitted. More precisely: the category Δ^-_2 is given by the graph

$$\Delta^-_2 \;\equiv\; \mathbb{1} \overset{\delta^1_1}{\underset{\delta^1_0}{\xleftarrow{\quad i^0_0 \quad}}} \mathbb{2} \overset{\delta^2_0}{\underset{\delta^2_2}{\xrightarrow{\quad \delta^2_1 \quad}}} \mathbb{3}$$

subject to equalities

$$i^0_0 \cdot \delta^1_0 = 1, \quad i^0_0 \cdot \delta^1_1 = 1, \quad \delta^2_0 \cdot \delta^1_1 = \delta^2_2 \cdot \delta^1_0, \quad \delta^2_1 \cdot \delta^1_0 = \delta^2_0 \cdot \delta^1_0, \quad \delta^2_1 \cdot \delta^1_1 = \delta^2_1 \cdot \delta^1_1.$$

We denote by $J^- : \Delta^-_2 \longrightarrow \mathsf{Pos}$ the inclusion.

Definition 3.11 (Lack 2002). A diagram $D : \Delta^{-op}_2 \longrightarrow \mathscr{X}$ is called a *coherence datum* in $\mathscr{X}$. The colimit $J^- * D$ is called a *quotient* of D.

Remark 3.12. The colimit $J^- * D$ of a coherence datum is called a *codescent* of D in Lack (2002). In our context, we prefer to call the colimit $J^- * D$ a *quotient* of D rather than a codescent of D.

Since every congruence is a coherence datum, the above definition can be applied to congruences. Thus

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Definition 3.13. The *quotient of a congruence* is the quotient of the underlying coherence datum.

Remark 3.14. Due to enrichment in posets, the *computation* of quotients of general coherence data reduces to the computation of coinserters of $D\delta_0^1$, $D\delta_1^1$. This follows from the general coherence conditions for a quotients (see Lack (2002), where quotients are called codescents), specialised to the case of enrichment over posets.

If a congruence happens to be an equivalence relation, then its quotient can be computed as a coequalizer.

Although the *computation* of quotients of congruences can be simplified, the *definition* of a congruence cannot be simplified. Observe that we need the full strength of the definition of a congruence in the proof of exactness of Pos, see Proposition 3.20. More in detail: congruences should be 'transitive' and this is exactly what the object A_2 and the morphism $d_1^2 : A_2 \longrightarrow A_1$ are responsible for. □

Definition 3.15. We say that a morphism is *effective* if it is a coinserter of some pair and that a congruence is *effective* if it is a kernel congruence.

Remark 3.16. Effective congruences are the ordered analogue of effective equivalence relations (Barr *et al.* 1971). Effective morphisms are called *P-regular* in Bloom and Wright (1983).

Lemma 3.17. Any effective morphism is an so-morphism.

Proof. Easy: use couniversality of a coinserter. The 1-dimensional aspect yields the required diagonal and the 2-dimensional aspect yields the 2-dimensional aspect of orthogonality. □

The above result establishes that 'every reg-epi is strong epi' for our factorisation system of so-morphisms and ff-morphisms. The gist of the definition of regularity is the converse of this statement. The gist of the definition of exactness is that 'congruences are precisely the kernel congruences.'

Definition 3.18. A category $\mathscr{X}$ is called *regular*, provided that the following four properties are satisfied:

(R1) $\mathscr{X}$ has finite limits.
(R2) $\mathscr{X}$ has $(\mathsf{so}, \mathsf{ff})$-factorisations.
(R3) so-morphisms are stable under pullbacks.
(R4) so-morphisms are exactly the effective morphisms.

If, in addition, $\mathscr{X}$ verifies the following condition

(Ex) Every congruence in $\mathscr{X}$ is effective, i.e. it is of the form $\ker(f)$,

then $\mathscr{X}$ is called *exact*.

Remark 3.19. Let us stress our convention: when we say a category, we mean a category enriched in posets. Categories that are not enriched, are called ordinary.

In Example 3.21 below we show that the *enriched* category Set is regular but not exact in the enriched sense, although the *ordinary* category Set is exact in the ordinary sense

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(see Barr *et al.* (1971)). The intuitive reason for this is that sets are closed under quotienting by equations, but discrete posets are not closed under quotienting by inequations.

Proposition 3.20 (Exactness of Pos **and presheaf categories).** Every category $[\mathscr{S}^{op}, \mathsf{Pos}]$, $\mathscr{S}$ small, is exact.

Proof. We prove exactness of Pos, exactness of $[\mathscr{S}^{op}, \mathsf{Pos}]$ follows by reasoning pointwise.

The only non-trivial condition to verify is (Ex). Suppose therefore that

$$\begin{tikzcd} \sim \equiv A_2 \arrow[r, shift left=2, "d_2^2"] \arrow[r, "d_1^2" description] \arrow[r, shift right=2, "d_0^2"'] & A_1 \arrow[r, shift left=2, "d_1^1"] \arrow[r, shift right=2, "d_0^1"'] & A_0 \arrow[l, "i_0^0" description] \end{tikzcd}$$

is a congruence on A_0. Form its quotient $q : A_0 \longrightarrow Q$ as in Example 2.4 and consider the kernel

$$\begin{tikzcd} \ker(q) \equiv P_2 \arrow[r, shift left=2, "p_2^2"] \arrow[r, "p_1^2" description] \arrow[r, shift right=2, "p_0^2"'] & q/q \arrow[r, shift left=2, "p_1^1"] \arrow[r, shift right=2, "p_0^1"'] & A_0 \arrow[l, "p_0^0" description] \end{tikzcd}$$

We claim that $\ker(q) = \sim$.

Denote by $z : A_1 \longrightarrow q/q$ the unique morphism such that the equality

$$\begin{tikzcd} A_1 \arrow[dr, "z"] & & \\ & q/q \arrow[r, "p_1^1"] \arrow[d, "p_0^1"'] \arrow[dr, phantom, "\nearrow"] & A_0 \arrow[d, "q"] \\ & A_0 \arrow[r, "q"'] & Q \end{tikzcd} \quad = \quad \begin{tikzcd} A_1 \arrow[r, "d_1^1"] \arrow[d, "d_0^1"'] \arrow[dr, phantom, "\nearrow"] & A_0 \arrow[d, "q"] \\ A_0 \arrow[r, "q"'] & Q \end{tikzcd}$$

holds, where the lax square on the left is a comma object (see Example 2.3).

In particular, the diagram

$$\begin{tikzcd} A_1 \arrow[r, "z"] \arrow[dr, "\langle d_0^1, d_1^1 \rangle"'] & q/q \arrow[d, "\langle p_0^1, p_1^1 \rangle"] \\ & A_0 \times A_0 \end{tikzcd}$$

commutes. It follows that z reflects order, since $\langle d_0^1, d_1^1 \rangle$ does ($\sim$ is a congruence). We need to prove that z is surjective. To that end, consider an object of q/q, i.e. a pair (a', a) such that $qa' \leqslant qa$. Use now the description of inequality in a quotient of Example 2.4 to find a finite sequence $f_0, \ldots, f_{n-1}$ of objects in A_1 such that the inequalities

$$a' \leqslant d_0^1(f_0), \quad d_1^1(f_0) \leqslant d_0^1(f_1), \quad d_1^1(f_1) \leqslant d_0^1(f_2), \quad \ldots, \quad d_1^1(f_{n-1}) \leqslant a,$$

hold in A_0.

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Using the fact that the span (d_0^1, A_0, d_1^1) is a two-sided discrete fibration, one can find a sequence $f_0^*, \dots, f_{n-1}^*$ of elements of A_1 such that the equalities

$$a' = d_0^1(f_0^*), \quad d_1^1(f_0^*) = d_0^1(f_1^*), \quad d_1^1(f_1^*) = d_0^1(f_2^*), \quad \dots, \quad d_1^1(f_{n-1}^*) = a,$$

hold in A_0. Since $\sim$ is a category object, the sequence $f_0^*, \dots, f_{n-1}^*$ composes (using d_1^2) to an element f^* of A_1 such that $a' = d_0^1(f^*)$ and $d_1^1(f^*) = a$. Hence, $z(f^*) = (a', a)$, and we proved that $z : A_1 \longrightarrow q/q$ is surjective.

Thus $q/q = A_1$, hence $A_2 = P_2$ by uniqueness of pullbacks. It remains to be proved that $d_1^2 = p_1^2$. But this follows easily.

We proved that $\ker(q) = \sim$, the proof of exactness of Pos is finished. □

Example 3.21 (The category Set (having discrete orders on hom-sets) is regular but not exact). Regularity of Set is easy: observe that the effective morphisms are precisely the epis (and these are precisely the surjective mappings).

We exhibit a congruence that is not effective. Consider the *truncated nerve*

$$\mathsf{nerve}(2) \equiv A_2 \; \begin{array}{c} \xrightarrow{d_2^2} \\ \xrightarrow{d_1^2} \\ \xrightarrow[d_0^2]{} \end{array} \; A_1 \; \begin{array}{c} \xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{} \end{array} \; A_0$$

of the two-element chain 2.

More in detail: A_0 is the two-element set $\{0, 1\}$, the set A_1 has as elements the pairs (i, j) with $i \leqslant j$ in 2, the set A_2 has as elements the triples (i, j, k) with $i \leqslant j \leqslant k$ in 2. All the connecting morphisms are defined in the obvious way.

It is easy to see that $\mathsf{nerve}(2)$ is a congruence. Yet there is no mapping $f : A_0 \longrightarrow X$ such that $\ker(f)$ would be $\mathsf{nerve}(2)$. Suppose the contrary. Since comma objects in Set reduce to pullbacks (the orders on hom-sets of Set are discrete), $\ker(f)$ is the following diagram

$$\ker(f) \equiv P_2 \; \begin{array}{c} \xrightarrow{d_2^2} \\ \xrightarrow{d_1^2} \\ \xrightarrow[d_0^2]{} \end{array} \; P_1 \; \begin{array}{c} \xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{} \end{array} \; A_0$$

and the set P_1 has either two elements $(0, 0)$ and $(1, 1)$ in case $f0 \neq f1$, or four elements $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ in case $f0 = f1$. But A_1 in $\mathsf{nerve}(2)$ has three elements: $(0, 0)$, $(0, 1)$ and $(1, 1)$. □

If we equip the A_i in the proof above with the operations of a (semi)lattice, the same argument shows that the categories of discretely ordered semilattices, or lattices, or distributive lattices are not exact. But note that if (semi)lattices are equipped with their natural order, then these categories are definable by inequalities and, as any variety, they are exact (Theorem 5.9).

As we are going to show now, the situation is the converse for Boolean algebras: Whereas lattices form a variety with their natural order but not with the discrete order, Boolean algebras form a variety with the discrete order but not with their natural order

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(for lattices discreteness is not definable by inequalities, whereas Boolean algebras are necessarily discrete).

Example 3.22 (The category of Boolean algebras (having discrete orders on hom-sets) is exact). We first show that hom-sets must be discretely ordered. To this end, assume that in some Boolean algebra we have elements $a \sqsubseteq b$ in some order $\sqsubseteq$ about which we only assume that it makes all operations monotone. It follows from $a \to a = b \to b = 1$ and implication being monotone that $a \to b = 1$ and $b \to a = 1$, hence $a = b$. In other words, the discrete order is the only order that makes all operations of a Boolean algebra monotone.

To show that Boolean algebras are exact, let

$$
\begin{tikzcd}
A_2 \arrow[r, shift left=2, "d_2^2"] \arrow[r, "d_1^2" description] \arrow[r, shift right=2, "d_0^2"'] & A_1 \arrow[r, shift left=2, "d_1^1"] \arrow[r, shift right=2, "d_0^1"'] & A_0 \arrow[l, "i_0^0" description]
\end{tikzcd}
$$

be a congruence (in the ordered sense) of Boolean algebras. Due to A_1 being an algebra, if $(a, b) \in A_1$ then imitating the reasoning of the previous paragraph, we also obtain $(b, a) \in A_1$. It follows that A_1 is equipped with an operation $s : A_1 \longrightarrow A_1$ making it into an equivalence relation (see Remark 3.5). But since Boolean algebras form an ordinary variety, we know that equivalence relations are effective. □

Remark 3.23.

1. The above proof works verbatim for Heyting algebras instead of Boolean algebras. Actually, for Boolean algebras the argument in the second paragraph can be made more succinct by noting that the morphism s is given by negation.
2. More generally, it will follow from Theorem 5.9 that any variety of ordinary algebras which can be equipped only with the discrete order is exact. A further example of this situation is given by the variety of groups.
3. The reason that Boolean algebras, Heyting algebras, and groups can only be discretely ordered is that the enriched categorical setting studied in this paper enforces operations to be monotone and does not allow us to have negation, implication, or inverse as non-monotone but antitone.

4. Some technical results

In this section, we gather some auxiliary results that we will use in Section 5:

1. We prove that the category $\mathsf{Cong}(\mathscr{X})$ of all congruences on an exact category $\mathscr{X}$ has all limits that $\mathscr{X}$ has.
2. We summarise properties of an adjunction $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ in case the counit $\varepsilon_A : FUA \longrightarrow A$ is an effective morphism (i.e. when it is a coinserter of some pair).
3. We prove that the category $\mathscr{X}^{\mathbb{T}}$ of Eilenberg–Moore algebras for a monad $\mathbb{T}$ is regular, whenever $\mathscr{X}$ is regular and the functor of the monad $\mathbb{T}$ preserves so-morphisms.

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4.1. *Limits of congruences*

We denote by $\mathsf{Cong}(\mathscr{X})$ the full subcategory of $[\Delta_2^{-op}, \mathscr{X}]$ spanned by congruences in $\mathscr{X}$. To be more specific: given coherence data

$$\mathbb{X} \equiv X_2 \;\begin{matrix} \xrightarrow{d_2^2} \\ \xrightarrow{d_1^2} \\ \xrightarrow[d_0^2]{} \end{matrix}\; X_1 \;\begin{matrix} \xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{} \end{matrix}\; X_0 \quad \text{and} \quad \mathbb{Y} \equiv Y_2 \;\begin{matrix} \xrightarrow{d_2^2} \\ \xrightarrow{d_1^2} \\ \xrightarrow[d_0^2]{} \end{matrix}\; Y_1 \;\begin{matrix} \xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{} \end{matrix}\; Y_0$$

then a *morphism* $f : \mathbb{X} \longrightarrow \mathbb{Y}$ is a triple $f_0 : X_0 \longrightarrow Y_0$, $f_1 : X_1 \longrightarrow Y_1$, $f_2 : X_2 \longrightarrow Y_2$ of morphisms in $\mathscr{X}$ making all the relevant squares commutative. Given morphisms $f, g : \mathbb{X} \longrightarrow \mathbb{Y}$, we put $f \leqslant g$ iff $f_i \leqslant g_i$ for all $i = 0, 1, 2$.

Lemma 4.1. Suppose $\mathscr{X}$ is exact. Then the category $\mathsf{Cong}(\mathscr{X})$ is reflective in $[\Delta_2^{-op}, \mathscr{X}]$. In particular, $\mathsf{Cong}(\mathscr{X})$ is closed in $[\Delta_2^{-op}, \mathscr{X}]$ under limits.

Proof. Suppose

$$\mathbb{X} \equiv X_2 \;\begin{matrix} \xrightarrow{d_2^2} \\ \xrightarrow{d_1^2} \\ \xrightarrow[d_0^2]{} \end{matrix}\; X_1 \;\begin{matrix} \xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{} \end{matrix}\; X_0$$

is a coherence datum. Define the congruence

$$\mathbb{X}^* \equiv X_2^* \;\begin{matrix} \xrightarrow{d_2^2} \\ \xrightarrow{d_1^2} \\ \xrightarrow[d_0^2]{} \end{matrix}\; X_1^* \;\begin{matrix} \xrightarrow{d_1^1} \\ \xleftarrow{i_0^0} \\ \xrightarrow[d_0^1]{} \end{matrix}\; X_0^*$$

as $\ker(q)$, where $q : X_0 \longrightarrow Q$ is the quotient of $\mathbb{X}$.

We claim that there is a morphism $e : \mathbb{X} \longrightarrow \mathbb{X}^*$ that is universal.

1. Definition of e.

   The morphism e has to be a natural transformation. Thus we define morphisms $e_0 : X_0 \longrightarrow X_0^*$, $e_1 : X_1 \longrightarrow X_1^*$, $e_2 : X_2 \longrightarrow X_2^*$, and prove that all the naturality squares commute.

   We put $e_0 = 1_{X_0}$, morphisms e_1, e_2 are defined using universal properties:

$$\begin{tikzcd}
 & & X_0^* \arrow[dr, "q"] & & & & X_0 \arrow[dr, "q"] & \\
X_1 \arrow[r, "e_1"] & X_1^* \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & \uparrow & Q & = & X_1 \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & \uparrow & Q \\
 & & X_0^* \arrow[ur, "q"'] & & & & X_0 \arrow[ur, "q"'] &
\end{tikzcd}$$

   and

$$\begin{tikzcd}
 & & X_0^* \arrow[dr, "d_0^1"] & & & & X_1 \arrow[r, "e_1"] & X_1^* \arrow[dr, "d_0^1"] & \\
X_2 \arrow[r, "e_2"] & X_2^* \arrow[ur, "d_2^2"] \arrow[dr, "d_0^2"'] & & X_0^* & = & X_2 \arrow[ur, "d_2^2"] \arrow[dr, "d_0^2"'] & & & X_0^* \\
 & & X_1^* \arrow[ur, "d_1^1"'] & & & & X_1 \arrow[r, "e_1"'] & X_1^* \arrow[ur, "d_1^1"'] &
\end{tikzcd}$$

   where we use the universal property of a comma square and a pullback, respectively. That $e : \mathbb{X} \longrightarrow \mathbb{X}^*$ is natural follows by straightforward computations.

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2. Universality of e.
   Given $f : \mathbb{X} \longrightarrow \mathbb{Y}$ where $\mathbb{Y}$ is a congruence, we define a unique $f^\sharp : \mathbb{X}^* \longrightarrow \mathbb{Y}$ extending f along e.
   Since $\mathscr{X}$ is exact, there is $z : Y_0 \longrightarrow K$ such that $\mathbb{Y} = \ker(z)$. Further, the existence of f yields $z^\sharp : Q \longrightarrow K$ such that the square

$$\begin{tikzcd}
X_0 \arrow[r, "q"] \arrow[d, "f_0"'] & Q \arrow[d, "z^\sharp"] \\
Y_0 \arrow[r, "z"'] & K
\end{tikzcd}$$

   commutes.
   We put $f_0^\sharp = f_0$, and $f_1^\sharp$, $f_2^\sharp$ are defined by universal properties:

$$\begin{tikzcd}
 & & Y_0^* \arrow[dr, "z"] & & & & X_0^* \arrow[dr, "q"] & & \\
X_1^* \arrow[r, "f_1^\sharp"] & Y_1^* \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & \uparrow & K & = & X_1^* \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & \uparrow & Q \arrow[r, "z^\sharp"] & K \\
 & & Y_0^* \arrow[ur, "z"'] & & & & X_0^* \arrow[ur, "q"'] & &
\end{tikzcd}$$

   and

$$\begin{tikzcd}
 & & Y_1^* \arrow[dr, "d_0^1"] & & & & X_1^* \arrow[dr, "d_0^1"] & & \\
X_2^* \arrow[r, "f_2^\sharp"] & Y_2^* \arrow[ur, "d_2^2"] \arrow[dr, "d_0^2"'] & & Y_0^* & = & X_2^* \arrow[ur, "d_2^2"] \arrow[dr, "d_0^2"'] & & X_0^* \arrow[r, "f_1^\sharp"] & Y_1^* \\
 & & Y_1^* \arrow[ur, "d_1^1"'] & & & & X_1^* \arrow[ur, "d_1^1"'] & &
\end{tikzcd}$$

   where we have used the universal property of a comma square and a pullback, respectively.
   The 2-dimensional aspect of universality of e is verified analogously, using the 2-dimensional aspects of universality of comma objects and pullbacks.

$\square$

Remark 4.2. Lemma 4.1 is the generalisation of the case of classical universal algebra: congruences form a complete lattice; meet of congruences is the intersection of the underlying relations; join of congruences is the congruence generated by the union of the underlying relations.

Indeed: $\mathsf{Cong}(\mathscr{X})$ is as (co)complete as $\mathscr{X}$. Reflectivity states that limits in $\mathsf{Cong}(\mathscr{X})$ are formed on the level of $[\Delta_2^{-op}, \mathscr{X}]$; whereas colimits in $\mathsf{Cong}(\mathscr{X})$ are the reflections of colimits in $[\Delta_2^{-op}, \mathscr{X}]$.

4.2. *Properties of $F \dashv U$ with an effective counit*

In Proposition 4.7 below we show that, when the counit of $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ is effective, then the underlying functor U has nice properties. The properties resemble the properties of adjunctions of descent type in ordinary category theory. In proving these results we were much inspired by arguments given by John Duskin in Duskin (1969) for the case of monadicity over set-like ordinary categories.

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We first prove an easy result on the interaction of U with ff-morphisms, which does not depend on U having a left-adjoint.

Lemma 4.3. Suppose that $\mathscr{A}$ has finite limits and $U : \mathscr{A} \longrightarrow \mathscr{X}$ preserves them. Then U preserves ff-morphisms. If, moreover, U is conservative (i.e. if U reflects isomorphisms), then U reflects ff-morphisms.

Proof. It is easy to see that $m : X \longrightarrow Y$ is an ff-morphism in $\mathscr{A}$ iff the canonical map $c(m) : 1_X/1_X \longrightarrow m/m$ between the comma objects is an isomorphism. Hence, U preserves ff-morphisms if U preserves comma objects.

If, moreover, U reflects isomorphisms, then U reflects ff-morphisms, by the same argument. □

For the proof of Proposition 4.7 we will need the following 'dual' of ff-morphisms.

Definition 4.4. We say that $e : A \longrightarrow B$ is a *co*-ff-*morphism* if it is ff in $\mathscr{X}^{op}$, or, equivalently, if $\mathscr{X}(e, Z) : \mathscr{X}(B, Z) \longrightarrow \mathscr{X}(A, Z)$ is order-reflecting, for every Z.

Remark 4.5. A *co*-ff-*morphism* is necessarily epi. Co-ff-morphisms are called P-epis in Bloom and Wright (1983), or absolutely dense in El Bashir and Velebil (2002).

We recall that when we say limit we mean limit in the sense of enriched categories.

Lemma 4.6. Suppose $\mathscr{X}$ has finite limits. Then every so-morphism is a co-ff-morphism.

Proof. Let $e : A \longrightarrow B$ be an so-morphism. Consider $u \cdot e \leqslant v \cdot e$ and form the inserter i of u and v. Consider the unique mediating map $k : A \longrightarrow E$ such that $i \cdot k = e$. Then the square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "k"'] & B \arrow[d, "1_B"] \\ E \arrow[r, "i"'] & B \end{tikzcd}$$

commutes. Since i is ff by its universal property, we can infer that i is a split epi, hence an isomorphism. Thus, $u \leqslant v$ and we proved that e is a co-ff-morphism. □

Proposition 4.7. Suppose $F \dashv U : \mathscr{A} \longrightarrow \mathscr{X}$ is an adjunction, such that every component ε_A of the counit is effective. Then, the following hold:

1. U is locally order-reflecting. That is, the monotone action $U_{A',A} : \mathscr{A}(A', A) \longrightarrow \mathscr{X}(UA', UA)$ of the functor U is order-reflecting, for every A', A.
2. U preserves and reflects congruences.
3. U preserves and reflects limits.
4. The comparison functor $K : \mathscr{A} \longrightarrow \mathscr{X}^{\mathbb{T}}$ is fully faithful.
5. If, moreover, $\mathscr{A}$ is regular, then U reflects effective morphisms.

Proof.

1. Every effective morphism is a co-ff-morphism (use couniversal property of coinserters for that).

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Hence, every ε_A is a co-ff-morphism. Since the diagram

$$\begin{tikzcd} \mathscr{A}(A',A) \arrow[r, "U_{A',A}"'] \arrow[rr, bend left, "\mathscr{A}(\varepsilon_{A'},A)"] & \mathsf{Pos}(UA',UA) \arrow[r, "\cong"'] & \mathscr{A}(FUA',A) \end{tikzcd}$$

commutes, the proof is finished.

2. Since U is a right adjoint, it preserves congruences. Indeed: suppose

$$\begin{tikzcd} A_2 \arrow[r, shift left=2, "d_2^2"] \arrow[r, "d_1^2" description] \arrow[r, shift right=2, "d_0^2"'] & A_1 \arrow[r, shift left=2, "d_1^1"] \arrow[r, shift right=2, "d_0^1"'] & A_0 \arrow[l, "d_0^0" description] \end{tikzcd}$$

is a congruence in $\mathscr{A}$.

Since U preserves (finite) limits, it preserves category objects. Thus

$$\begin{tikzcd} UA_2 \arrow[r, shift left=2, "Ud_2^2"] \arrow[r, "Ud_1^2" description] \arrow[r, shift right=2, "Ud_0^2"'] & UA_1 \arrow[r, shift left=2, "Ud_1^1"] \arrow[r, shift right=2, "Ud_0^1"'] & UA_0 \arrow[l, "Ud_0^0" description] \end{tikzcd}$$

is a category object in $\mathscr{X}$.

By the same argument $U\langle d_0^1, d_1^1\rangle \cong \langle Ud_0^1, Ud_1^1\rangle$. Since U preserves ff-morphisms (being a right adjoint, see Lemma 4.3), we proved that $\langle Ud_0^1, Ud_1^1\rangle$ is an ff-morphism.

Since being a two-sided discrete fibration is a representable notion, see Remark 3.7, the isomorphisms

$$\mathscr{X}(X, \langle Ud_0^1, Ud_1^1\rangle) \cong \mathscr{X}(X, U\langle d_0^1, d_1^1\rangle) \cong \mathscr{X}(FX, \langle d_0^1, d_1^1\rangle)$$

prove that the span $(\mathscr{X}(X, Ud_0^1)\mathscr{X}(X, UA_1), \mathscr{X}(X, Ud_1^1))$ is a two-sided discrete fibration, for any X. Hence, the span (Ud_0^1, UA_1, Ud_1^1) is a two-sided discrete fibration.

For the reflection of congruences, consider a coherence datum $\mathbb{D} : \Delta_2^{-op} \longrightarrow \mathscr{A}$ such that the composite $U\mathbb{D} : \Delta_2^{-op} \longrightarrow \mathscr{X}$ is a congruence. To prove that $\mathbb{D}$ is a congruence, we need to prove that the composite $\mathscr{A}(A,\mathbb{D}) : \Delta_2^{-op} \longrightarrow \mathsf{Pos}$ is a congruence, for every A. Observe that every $\mathscr{A}(FX,\mathbb{D})$ is a congruence, since $\mathscr{A}(FX,\mathbb{D}) \cong \mathscr{X}(X, U\mathbb{D})$ holds.

Thus it suffices to present $\mathscr{A}(A,\mathbb{D})$ as a limit of congruences in Pos and then use Lemma 4.1.

Since ε_A is assumed to be effective, there is a a coinserter of the form

$$\begin{tikzcd} & FUA \arrow[dr, "\varepsilon_A"] \arrow[dd, phantom, "\uparrow"] & \\ A_1 \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & & A \\ & FUA \arrow[ur, "\varepsilon_A"'] & \end{tikzcd}$$

We claim that the pasting

$$\begin{tikzcd} & & FUA \arrow[dr, "\varepsilon_A"] \arrow[dd, phantom, "\uparrow"] & \\ FUA_1 \arrow[r, "\varepsilon_{A_1}"] & A_1 \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & & A \\ & & FUA \arrow[ur, "\varepsilon_A"'] & \end{tikzcd}$$

is a coinserter diagram. That is easy: ε_{A_1} is a co-ff-morphism, hence coinserter 'cocones' for d_0^1, d_1^1 coincide with coinserter 'cocones' for $d_0^1 \cdot \varepsilon_{A_1}$, $d_1^1 \cdot \varepsilon_{A_1}$.
Therefore, we have an *inserter* diagram

$$\begin{tikzcd} & \mathscr{A}(FUA, \mathbb{D}) \arrow[dr, "\mathscr{A}(d_1^1 \varepsilon_{A_1}, \mathbb{D})"] & \\ \mathscr{A}(A, \mathbb{D}) \arrow[ur, "\mathscr{A}(\varepsilon_A, \mathbb{D})"] \arrow[dr, "\mathscr{A}(\varepsilon_A, \mathbb{D})"'] & \uparrow & \mathscr{A}(FUA_1, \mathbb{D}) \\ & \mathscr{A}(FUA, \mathbb{D}) \arrow[ur, "\mathscr{A}(d_0^1 \varepsilon_{A_1}, \mathbb{D})"'] & \end{tikzcd}$$

in $[\Delta_2^{-op}, \mathsf{Pos}]$. But both $\mathscr{A}(FUA_1, \mathbb{D})$ and $\mathscr{A}(FUA, \mathbb{D})$ are congruences. By Lemma 4.1, $\mathscr{A}(A, \mathbb{D})$ is a congruence.

3. U preserves limits since it is a right adjoint.
For reflecting limits, consider a diagram $D : \mathscr{D} \longrightarrow \mathscr{A}$ and a weight $W : \mathscr{D} \longrightarrow \mathsf{Pos}$. Suppose $\gamma : W \longrightarrow \mathscr{A}(A, D-)$ is a cylinder such that the composite

$$\overline{\gamma} \equiv W \xrightarrow{\gamma} \mathscr{A}(A, D-) \xrightarrow{U_{A,D-}} \mathscr{X}(UA, UD-)$$

is a limit cylinder in $\mathscr{X}$. This means that the monotone map

$$\overline{\varphi}_X : \mathscr{X}(X, UA) \longrightarrow [\mathscr{D}, \mathsf{Pos}](W, \mathscr{X}(X, UD-)), \quad f \mapsto \mathscr{X}(f, -) \cdot \overline{\gamma}$$

is an isomorphism, natural in X.
We need to prove that the monotone map

$$\varphi_{A'} : \mathscr{A}(A', A) \longrightarrow [\mathscr{D}, \mathsf{Pos}](W, \mathscr{A}(A', D-)), \quad f \mapsto \mathscr{A}(f, -) \cdot \gamma$$

is an isomorphism, naturally in A'.
We will use a similar trick to (2) above. For observe that φ_{FX} is an isomorphism for every X: this follows from the commutative square

$$\begin{tikzcd} \mathscr{A}(FX, A) \arrow[r, "\varphi_{FX}"] \arrow[d, "\cong"'] & {[\mathscr{D}, \mathsf{Pos}](W, \mathscr{A}(FX, D-))} \arrow[d, "\cong"] \\ \mathscr{X}(X, UA) \arrow[r, "\overline{\varphi}_X"'] & {[\mathscr{D}, \mathsf{Pos}](W, \mathscr{X}(X, UD-))} \end{tikzcd}$$

where the vertical maps are given by the adjunction bijections.
Expressing $\varepsilon_{A'}$ as a coinserter

$$\begin{tikzcd} & FUA' \arrow[dr, "\varepsilon_{A'}"] & \\ FUA'_1 \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & \uparrow & A' \\ & FUA' \arrow[ur, "\varepsilon_{A'}"'] & \end{tikzcd}$$

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in the same way as in (2) above, we see that both

$$
\begin{tikzcd}
 & \mathscr{A}(FUA',A) \arrow[dr, "\mathscr{A}(d_1^1,A)"] & \\
\mathscr{A}(A',A) \arrow[ur, "\mathscr{A}(\varepsilon_{A'},A)"] \arrow[dr, "\mathscr{A}(\varepsilon_{A'},A)"'] & \uparrow & \mathscr{A}(FUA'_1,A) \\
 & \mathscr{A}(FUA',A) \arrow[ur, "\mathscr{A}(d_0^1,A)"'] &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & {[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(FUA',D-)} \arrow[dr, "{[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(d_1^1,D-))}"] & \\
{[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(A',D-)} \arrow[ur, "{[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(\varepsilon_{A'},D-))}"] \arrow[dr, "{[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(\varepsilon_{A'},D-))}"'] & \uparrow & {[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(FUA'_1,D-)} \\
 & {[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(FUA',D-)} \arrow[ur, "{[\mathscr{D},\mathsf{Pos}](W,\mathscr{A}(d_0^1,D-))}"'] &
\end{tikzcd}
$$

are inserters of isomorphic diagrams. Thus φ_A is an isomorphism by the essential uniqueness of inserters.

4. Since $U = U^{\mathbb{T}} \cdot K$, the functor K is order-reflecting. In particular, K is faithful.
We prove that the functor K is full. To that end, consider $f : KA \longrightarrow KB$. Thus suppose the square

$$
\begin{tikzcd}
UFUA \arrow[r, "UFf"] \arrow[d, "U\varepsilon_A"'] & UFUB \arrow[d, "U\varepsilon_B"] \\
UA \arrow[r, "f"'] & UB
\end{tikzcd}
$$

commutes.
Since $\varepsilon_A : FUA \longrightarrow A$ is effective, there is a coinserter

$$
\begin{tikzcd}
 & FUA \arrow[dr, "\varepsilon_A"] & \\
A_1 \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & \uparrow & A \\
 & FUA \arrow[ur, "\varepsilon_A"'] &
\end{tikzcd}
$$

To prove that

$$
\begin{tikzcd}
 & FUA \arrow[dr, "\varepsilon_B \cdot Ff"] & \\
A_1 \arrow[ur, "d_1^1"] \arrow[dr, "d_0^1"'] & \uparrow & B \\
 & FUA \arrow[ur, "\varepsilon_B \cdot Ff"'] &
\end{tikzcd}
$$

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consider first the pasting

$$
\begin{tikzcd}
 & UFUA \arrow[r, "UFf"] \arrow[dr, "U\varepsilon_A"] & UFUB \arrow[dr, "U\varepsilon_B"] & \\
UA_1 \arrow[ur, "Ud_1^1"] \arrow[dr, "Ud_0^1"'] \arrow[rr, phantom, "\uparrow"] & & UA \arrow[r, "f"] & UB \\
 & UFUA \arrow[r, "UFf"'] \arrow[ur, "U\varepsilon_A"'] & UFUB \arrow[ur, "U\varepsilon_B"'] &
\end{tikzcd}
$$

and then use that U is locally order-reflecting.

By the universal property of coinserters there is a unique $h : A \longrightarrow B$ such that the square

$$
\begin{tikzcd}
FUA \arrow[r, "Ff"] \arrow[d, "\varepsilon_A"'] & FUB \arrow[d, "\varepsilon_B"] \\
A \arrow[r, "h"'] & B
\end{tikzcd}
$$

commutes. Therefore, $Uh \cdot U\varepsilon_A = f \cdot U\varepsilon_A$ holds (both are equal to $U\varepsilon_B \cdot UFf$). Since $U\varepsilon_A$ is epi, $Uh = f$ follows. Hence, K is full.

5. To prove the last assertion, suppose $e : A \longrightarrow B$ is such that Ue is effective in $\mathscr{X}$. Then FUe is effective. Thus in the naturality square

$$
\begin{tikzcd}
FUA \arrow[r, "FUe"] \arrow[d, "\varepsilon_A"'] & FUB \arrow[d, "\varepsilon_B"] \\
A \arrow[r, "e"'] & B
\end{tikzcd}
$$

the passage first-right-then-down is an so-morphism (use that every effective morphism is an so-morphism in $\mathscr{A}$). Therefore, e is an so-morphism, hence effective.

The proof is finished. □

4.3. *Regularity of $\mathscr{X}^{\mathbb{T}}$*

If an ordinary monad $\mathbb{T}$ on an ordinary category $\mathscr{X}$ preserves regular epis, then the category $\mathscr{X}^{\mathbb{T}}$ of algebras for the monad $\mathbb{T}$ is regular, see Barr *et al.* (1971). This result extends to the ordered setting. The proof, following the same lines as Barr *et al.* (1971), is presented below.

Example 4.8 (Bloom and Wright (1983), Section 8, Example 5). Consider the adjunction $- \bullet \mathbb{2} \dashv [\mathbb{2}, -] : \mathsf{Pos} \longrightarrow \mathsf{Pos}$. The resulting monad $\mathbb{T} = (T, \eta, \mu)$ on Pos does not preserve the so-morphism $e : 2 \longrightarrow \mathbb{2}$.

We will need the following technical notion.

Definition 4.9 (Kurz and Velebil 2013). Suppose $U : \mathscr{A} \longrightarrow \mathscr{X}$ is any functor. We say that $f : A \longrightarrow B$ is U-*final* if the following commutative diagram

$$\begin{tikzcd} \mathscr{A}(B,B') \arrow[r, "\mathscr{A}(f,B')"] \arrow[d, "U_{B,B'}"'] & \mathscr{A}(A,B') \arrow[d, "U_{A,B'}"] \\ \mathscr{X}(UB,UB') \arrow[r, "\mathscr{X}(Uf,UB')"'] & \mathscr{X}(UA,UB') \end{tikzcd}$$

is a pullback, for every B'.

Remark 4.10. Thus, as expected, U-finality has two aspects:

1. For every $g : UB \longrightarrow UB'$, if $g \cdot Uf$ is of the form Uh, then there is a unique $g' : B \longrightarrow B'$ such that $Ug' = g$.
2. If $g_1 \leqslant g_2 : UB \longrightarrow UB'$ and if $g_1 \cdot Uf \leqslant g_2 \cdot Uf$ has the form $Uh_1 \leqslant Uh_2$, then $g_1' \leqslant g_2'$.

Lemma 4.11. Suppose $\mathscr{X}$ has finite limits and $\mathbb{T}$ is a monad on $\mathscr{X}$ that preserves so-morphisms. If $U^{\mathbb{T}}e : A \longrightarrow B$ is an so-morphism in $\mathscr{X}$, then $e : (A,a) \longrightarrow (B,b)$ is $U^{\mathbb{T}}$-final.

Proof. Consider $f : B \longrightarrow U^{\mathbb{T}}(C,c)$, such that the diagram

$$\begin{tikzcd} TA \arrow[r, "Te"] \arrow[d, "a"'] & TB \arrow[r, "Tf"] \arrow[d, "b"'] & TC \arrow[d, "c"] \\ A \arrow[r, "e"'] & B \arrow[r, "f"'] & C \end{tikzcd}$$

commutes. The morphism $Te : TA \longrightarrow TB$ is so, hence epi by Lemma 4.6. Thus $f : (B,b) \longrightarrow (C,c)$ is a $\mathbb{T}$-algebra morphism.

The 2-dimensional aspect of finality follows analogously, using the fact that Te is a co-ff-morphism by Lemma 4.6. □

Proposition 4.12. Suppose $\mathscr{X}$ has finite limits and let $\mathbb{T}$ be a monad that preserves so-morphisms. Then $U^{\mathbb{T}} : \mathscr{X}^{\mathbb{T}} \longrightarrow \mathscr{X}$ reflects so-morphisms. If $\mathscr{X}$ has $(\mathsf{so},\mathsf{ff})$-factorisations, $U^{\mathbb{T}}$ preserves so-morphisms.

Proof. Suppose $e : (A,a) \longrightarrow (B,b)$ is a $\mathbb{T}$-algebra morphism such that $U^{\mathbb{T}}e = e : A \longrightarrow B$ is an so-morphism. Consider a commutative square

$$\begin{tikzcd} (A,a) \arrow[r, "e"] \arrow[d, "u"'] & (B,b) \arrow[d, "v"] \\ (X,x) \arrow[r, "m"'] & (Y,y) \end{tikzcd}$$

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with m an ff-morphism in $\mathscr{X}^{\mathbb{T}}$. Since $U^{\mathbb{T}}$ preserves and reflects ff-morphisms by Lemma 4.3, the square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

has a unique diagonal fill-in $d : B \longrightarrow X$ that is a $\mathbb{T}$-algebra morphism by $U^{\mathbb{T}}$-finality. This proves that $U^{\mathbb{T}}$ reflects so-morphisms.

The preservation: consider an so-morphism $e : (A, a) \longrightarrow (B, b)$. Form the $(\mathsf{so}, \mathsf{ff})$-factorisation $m \cdot e'$ of $U^{\mathbb{T}} e$. Then the diagram

$$\begin{tikzcd} TA \arrow[r, "Te'"'] \arrow[d, "a"'] \arrow[rr, bend left, "Te"] & TA' \arrow[r, "Tm"'] \arrow[d, dotted, "a'"'] & TB \arrow[d, "b"] \\ A \arrow[r, "e'"] \arrow[rr, bend right, "e"'] & A' \arrow[r, "m"] & B \end{tikzcd}$$

commutes and there is a diagonal fill-in $a' : TA' \longrightarrow A'$ as indicated, since Te' is an so-morphism. The pair (A', a') is a $\mathbb{T}$-algebra, since m is a monomorphism. Thus we have $e = m \cdot e'$ in $\mathscr{X}^{\mathbb{T}}$. But e is so and m is ff (by Lemma 4.3). Therefore, m is an isomorphism and we have proved that $e = e'$. Thus $U^{\mathbb{T}}$ reflects so-morphisms. $\square$

Corollary 4.13. Suppose that $\mathscr{X}$ has finite limits and $(\mathsf{so}, \mathsf{ff})$-factorisations. Suppose further that $\mathbb{T}$ is any monad on $\mathscr{X}$. Then the following are equivalent:

1. $\mathbb{T}$ preserves so-morphisms.
2. $U^{\mathbb{T}}$ preserves so-morphisms.

Proof. By Proposition 4.12 it suffices to prove that (2) implies (1). Suppose that $e : A \longrightarrow B$ is an so-morphism. We prove that $Te : (TA, \mu_A) \longrightarrow (TB, \mu_B)$ is an so-morphism in $\mathscr{X}^{\mathbb{T}}$. To that end, consider the square

$$\begin{tikzcd} (TA, \mu_A) \arrow[r, "Te"] \arrow[d, "u"'] & (TB, \mu_B) \arrow[d, "v"] \\ (X, x) \arrow[r, "m"'] & (Y, y) \end{tikzcd}$$

with m an ff-morphism in $\mathscr{X}^{\mathbb{T}}$.

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Then the square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "\eta_A"'] & B \arrow[d, "\eta_B"] \arrow[ddl, dotted, "d"] \\ TA \arrow[d, "u"'] & TB \arrow[d, "v"] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

commutes in $\mathscr{X}$ and the transpose $d^\sharp : (TB, \mu_B) \longrightarrow (X, x)$ under $F^{\mathbb{T}} \dashv U^{\mathbb{T}}$ of the unique diagonal d proves the 1-dimensional aspect of Te being an so-morphism. The 2-dimensional aspect is proved analogously.

Since $Te : (TA, \mu_B) \longrightarrow (TB, \mu_B)$ is an so-morphism in $\mathscr{X}^{\mathbb{T}}$, so is $U^{\mathbb{T}}Te = Te : TA \longrightarrow TB$. □

Corollary 4.14. Suppose $\mathscr{X}$ is regular and $\mathbb{T}$ is a monad preserving so-morphisms. Then $\mathscr{X}^{\mathbb{T}}$ is regular.

Proof. $\mathscr{X}^{\mathbb{T}}$ has finite limits since $\mathscr{X}$ has them and $U^{\mathbb{T}}$ creates limits. Proposition 4.12 and Lemma 4.3 prove that $(\mathsf{so}, \mathsf{ff})$-factorisations exist in $\mathscr{X}^{\mathbb{T}}$. Moreover, so-morphisms in $\mathscr{X}^{\mathbb{T}}$ are pullback stable, since $U^{\mathbb{T}}$ preserves pullbacks, and preserves and reflects so-morphisms.

It remains to be proved that so-morphisms of $\mathscr{X}^{\mathbb{T}}$ are exactly the quotients of congruences in $\mathscr{X}^{\mathbb{T}}$. By Lemma 3.17 it suffices to prove that every so-morphism in $\mathscr{X}^{\mathbb{T}}$ is effective.

Consider an so-morphism $e : (A, a) \longrightarrow (B, b)$ and form its kernel congruence $\ker(e)$. By Proposition 4.7 $U^{\mathbb{T}} \ker(e)$ is a congruence and it is easy to see that $U^{\mathbb{T}} \ker(e) = \ker(U^{\mathbb{T}} e)$. Hence, $U^{\mathbb{T}} e$ is a quotient of $U^{\mathbb{T}} \ker(e)$, since $\mathscr{X}$ is regular. Now use $U^{\mathbb{T}}$-finality of $e : (A, a) \longrightarrow (B, b)$ to conclude that e is a quotient of $\ker(e)$. □

5. Quasivarieties and varieties

In this section, we prove our main results (Theorems 5.9 and 5.13 below) that characterise varieties and quasivarieties of ordered algebras for signatures in the sense of Bloom and Wright (1983).

We start with precise definitions of signatures and their algebras.

Definition 5.1. Let λ be a regular cardinal. Denote by $|\mathsf{Set}_\lambda|$ the discrete category having sets of cardinality less than λ as objects. A *λ-ary signature* Σ is a functor $\Sigma : |\mathsf{Set}_\lambda| \longrightarrow \mathsf{Pos}$.

Thus, a signature is a collection $(\Sigma n)_n$ of posets, indexed by sets of cardinalities smaller than λ. The elements of the poset Σn are called *n-ary operations*.

Definition 5.2. Given a λ-ary signature Σ, we denote by $H_\Sigma : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ the corresponding *polynomial functor*, defined by

$$H_\Sigma X = \coprod_n X^n \bullet \Sigma n$$

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where the coproduct ranges over sets of cardinality less than λ. A category $\mathsf{Alg}(H_\Sigma)$ of Σ-*algebras and their homomorphisms* is the category of algebras for the functor H_Σ and algebra homomorphisms.

In more detail, a Σ-algebra is a morphism $a : H_\Sigma X \longrightarrow X$ in Pos. Due to the definition of H_Σ, to give a amounts to giving a collection $a_n : X^n \bullet \Sigma n \longrightarrow X$ of monotone maps, indexed by sets n of cardinality less than λ. Each a_n yields, due to the definition of a coproduct, a monotone mapping $[\![-]\!]_n^X : \Sigma n \longrightarrow \mathsf{Pos}(X^n, X)$. Thus it is convenient to think of a Σ-algebra as of a pair $(X, [\![-]\!]_n^X)$ consisting of a poset X and monotone maps $[\![\sigma]\!]_n^X : X^n \longrightarrow X$ for every σ in Σn. If $\sigma \leqslant \tau$ in Σn, then there is an inequality $[\![\sigma]\!]_n \leqslant [\![\tau]\!]_n$ in the poset $\mathsf{Pos}(X^n, X)$. When there is no confusion likely, we will omit the indices n and X in $[\![-]\!]_n^X$.

A homomorphism of algebras is a monotone map h making the square

$$\begin{tikzcd} H_\Sigma X \arrow[r, "H_\Sigma h"] \arrow[d, "a"'] & H_\Sigma Y \arrow[d, "b"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

commutative. By reasoning similar to the above, a monotone map $h : X \longrightarrow Y$ is a homomorphism iff the equality

$$h([\![\sigma]\!]_n^X(x_i)) = [\![\sigma]\!]_n^Y(hx_i)$$

holds for all n and all n-tuples (x_i) of elements of X.

Definition 5.3. Suppose that Σ is a λ-ary signature. We say that

1. $\mathscr{A}$ is a λ-ary *quasivariety* if $\mathscr{A}$ is equivalent to a full subcategory of $\mathsf{Alg}(H_\Sigma)$, defined by implications of the form

$$\bigwedge_{i \in I} (s'_i(x_{ij}) \sqsubseteq s_i(x_{ij})) \Rightarrow t'(x_k) \sqsubseteq t(x_k)$$

   where the cardinality of I is smaller than λ.
2. $\mathscr{A}$ is a λ-ary *variety* if $\mathscr{A}$ is equivalent to a full subcategory of $\mathsf{Alg}(H_\Sigma)$, defined by inequalities of the form

$$t'(x_k) \sqsubseteq t(x_k)$$

Remark 5.4. Since we are dealing with λ-ary signatures, one expects that λ-*filtered* colimits will play a prominent rôle. This is indeed the case: we only stress that all the notions concerning λ-filtered colimits are those that are appropriate for category theory enriched in posets.

We briefly recall the basic notions of the theory of λ-filtered colimits (and specialise them for the enrichment in posets). For details, see Max Kelly's paper (Kelly 1982). Notice that the phrasing and results are the same as in the case of ordinary categories, see Gabriel and Ulmer (1971) or Adámek and Rosický (1994).

1. By a *λ-filtered* colimit in $\mathscr{X}$ we mean a *conical* colimit of an ordinary functor $D : \mathscr{D} \longrightarrow \mathscr{X}_o$, where $\mathscr{D}$ is a λ-filtered ordinary category and $\mathscr{X}_o$ denotes the underlying ordinary category of $\mathscr{X}$.
   Here, by a *conical* colimit of $D : \mathscr{D} \longrightarrow \mathscr{X}_o$ we understand a colimit weighted by the functor that is constantly the one-element poset.
2. A functor $F : \mathscr{A} \longrightarrow \mathscr{B}$ is called *λ-accessible* if $\mathscr{A}$ has λ-filtered colimits and F preserves them.
3. An object X is called *λ-presentable* if the hom-functor $\mathscr{X}(X,-) : \mathscr{X} \longrightarrow \mathsf{Pos}$ is λ-accessible.
4. A category $\mathscr{X}$ is called *locally λ-presentable* if $\mathscr{X}$ is cocomplete and there is a small full dense subcategory $E : \mathscr{X}_\lambda \longrightarrow \mathscr{X}$ representing all λ-presentable objects of $\mathscr{X}$.

As examples of locally λ-presentable categories serve: the category Pos, every category of the form $[\mathscr{A}, \mathsf{Pos}]$ where $\mathscr{A}$ is small, every category of the form $\mathscr{X}^{\mathbb{T}}$ where $\mathscr{X}$ is locally λ-presentable and $\mathbb{T}$ is a λ-accessible monad (i.e. one, whose underlying functor T is λ-accessible). See Kelly (1982) and Bird (1984).

Every (quasi)variety $\mathscr{A}$ is equipped by a functor $U : \mathscr{A} \longrightarrow \mathsf{Pos}$ that arises as the composite of the fully faithful functor $K : \mathscr{A} \longrightarrow \mathsf{Pos}^{H_\Sigma}$ and the λ-accessible monadic functor $U^\Sigma : \mathsf{Pos}^{H_\Sigma} \longrightarrow \mathsf{Pos}$.

Lemma 5.5. Let $\mathscr{A}$ be a λ-ary quasivariety. Then $\mathscr{A}$ has λ-filtered colimits and the full inclusion $K : \mathscr{A} \longrightarrow \mathsf{Pos}^{H_\Sigma}$ preserves them.

Proof. It suffices to prove that if $(C, [\![-]\!])$ is a conical λ-filtered colimit of Σ-algebras $(Dd, [\![-]\!]^d)$ satisfying an implication

$$\bigwedge_{i \in I} (s'_i(x_{ij}) \sqsubseteq s_i(x_{ij})) \Rightarrow t'(x_k) \sqsubseteq t(x_k),$$

then $(C, [\![-]\!])$ satisfies this implication. Suppose therefore that $[\![(s'_i(x_{ij})]\!] \leqslant [\![s_i(x_{ij})]\!]$ holds in $(C, [\![-]\!])$, for all i. Since the diagram of $(Dd, [\![-]\!]^d)$'s is λ-filtered and since Σ is a λ-ary signature, there is d_0 such that $[\![(s'_i(x_{ij})]\!]^{d_0} \leqslant [\![s_i(x_{ij})]\!]^{d_0}$ holds in $(Dd_0, [\![-]\!]^{d_0})$. Therefore, $t'(x_k) \sqsubseteq t(x_k)$ holds in $(Dd_0, [\![-]\!]^{d_0})$. Using monotonicity of the colimit injections, $t'(x_k) \sqsubseteq t(x_k)$ holds in $(C, [\![-]\!])$. □

Thus, we can work with λ-ary (quasi)varieties as categories equipped with a λ-accessible functor into Pos. Using this observation, we can reformulate the main result of Bloom and Wright (1983) as follows:

Theorem 5.6 (The main theorem of Bloom and Wright (1983)). Suppose $U : \mathscr{A} \longrightarrow \mathsf{Pos}$ is a λ-accessible functor. Then U exhibits $\mathscr{A}$ as a λ-ary quasivariety iff the following conditions:

(Q1) $\mathscr{A}$ has coinserters.
(Q2) The action of U on hom-posets is order reflecting.
(Q3) U has a left adjoint F.

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(Q4) U preserves and reflects effective morphisms.
(Q5) U reflects isomorphisms.

are satisfied.

The functor U exhibits $\mathscr{A}$ as a variety for a bounded signature iff, in addition, the condition

(V) U reflects effective congruences.

holds.

Definition 5.7. An object P of a cocomplete category $\mathscr{A}$ is called a *λ-algebraic generator*, if it satisfies the following three properties:

1. Tensors $X \bullet P$ exist for every poset X.
2. P is a λ-presentable object in $\mathscr{A}$.
3. P is projective w.r.t. so-morphisms.
4. P is an so-generator, i.e. the canonical $\varepsilon_A : \mathscr{A}(P, A) \bullet P \longrightarrow A$ is an so-morphism.

Example 5.8. In any λ-ary quasivariety the free algebra on one generator is a λ-algebraic generator. For example, the one-element set 1 is a free algebra on one generator in the finitary quasivariety Set. Thus 1 is a finitary algebraic generator in Set. For a poset X, the tensor $X \bullet 1$ is the discrete poset of the connected components of X.

Our first intrinsic characterisation concerns varieties of ordered algebras. Compare the phrasing with Corollary 5.13 of Duskin (1969) and Proposition 3.2 of Vitale (1994).

Theorem 5.9 (Intrinsic characterisation of λ-ary varieties). For $\mathscr{A}$, the following are equivalent:

1. There is a λ-accessible functor $U : \mathscr{A} \longrightarrow \mathsf{Pos}$, exhibiting $(\mathscr{A}, U)$ as a λ-ary variety.
2. $\mathscr{A}$ is exact and there is an equivalence $\mathscr{A} \simeq \mathsf{Pos}^{\mathbb{T}}$, for a λ-accessible monad $\mathbb{T}$ on Pos.
3. $\mathscr{A}$ is exact, has coinserters, and possesses a λ-algebraic generator.

Proof.

1. implies 2. By Bloom and Wright (1983, Section 6, Lemma 4), $U : \mathscr{A} \longrightarrow \mathsf{Pos}$ is a λ-accessible monadic functor. Hence, $\mathscr{A} \simeq \mathsf{Pos}^{\mathbb{T}}$ for the λ-accessible monad $\mathbb{T}$ given by U. Since U preserves so-morphisms, the category $\mathscr{A}$ is regular by Corollary 4.14. Since U reflects effective congruences, $\mathscr{A}$ is exact.

2. implies 3. Assume $\mathscr{A} = \mathsf{Pos}^{\mathbb{T}}$. Then $\mathscr{A}$ is a locally λ-presentable category by Bird (1984, Theorem 6.9). Thus $\mathscr{A}$ has coinserters.
To conclude the proof, put P to be the free algebra $F\mathbb{1}$ on the one-element poset. We prove that P is a λ-algebraic generator.
    a. The tensor $X \bullet P$ is isomorphic to FX.
    b. The functor $U^{\mathbb{T}} \cong \mathscr{A}(P, -)$ is λ-accessible, hence P is λ-presentable.
    c. Since $U^{\mathbb{T}} \cong \mathscr{A}(P, -)$ holds, $\mathscr{A}(P, -)$ preserves so-morphisms by Corollary 4.13. This means precisely that P is so-projective.
    d. We only need to show that the counit ε_A of $F \dashv U$ is an so-morphism. But this is trivial: $U\varepsilon_A$ is a split epimorphism, hence an so-morphism in Pos. The monadic

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functor $U^{\mathbb{T}} : \mathsf{Pos}^{\mathbb{T}} \longrightarrow \mathsf{Pos}$ reflects so-morphisms, since $\mathbb{T}$ preserves so-morphisms by Proposition 4.12.

3. implies 1. Let P denote the λ-algebraic generator. Define $U = \mathscr{A}(P, -)$. Then U is λ-accessible, since P is λ-presentable. We verify conditions (Q1)–(Q5) and (V) for the pair $(\mathscr{A}, U)$.

(Q1) $\mathscr{A}$ has coinserters.
Trivial.

(Q3) U has a left adjoint.
Easy: $F \cong - \bullet P$.

(Q2) U is locally order-reflecting.
Since P is an so-generator, the counit ε_A of $F \dashv U$ is an so-morphism. Since $\mathscr{A}$ has finite limits (being exact), every so-morphism is a co-ff-morphism, see Lemma 4.6. Thus every $\mathscr{A}(\varepsilon_{A'}, A) \cong U_{A',A}$ is order-reflecting.

(Q4) U preserves and reflects effective morphisms.
Every effective morphism in $\mathscr{A}$ is an so-morphism. But U preserves so-morphisms, since P is so-projective. And every so-morphism in Pos is effective.
U reflects effective morphisms by Proposition 4.7.

(Q5) U reflects isomorphisms.
Suppose $f : A \longrightarrow B$ is such that Uf is an isomorphism. Since $\varepsilon_A : FUA \longrightarrow A$ and $\varepsilon_B : FUB \longrightarrow B$ are so-morphisms, the naturality square

$$\begin{tikzcd} FUA \arrow[r, "FUf"] \arrow[d, "\varepsilon_A"'] & FUB \arrow[d, "\varepsilon_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

tells us that f is an so-morphism.
We prove that f is an ff-morphism. To that end, consider an inequality $f \cdot u \leqslant f \cdot u$. Since Uf is an isomorphism, $Uu \leqslant Uv$ holds. And $u \leqslant v$ holds by (Q2).

(V) U reflects effective congruences.
Use Proposition 4.7 and the fact that $\mathscr{A}$ is exact.

□

Next we give a fundamental example of a category which is a quasivariety but not a variety.

Example 5.10 (The category Set is not a variety of ordered algebras). Recall that from Example 1 the finitary monadic discrete-poset functor $U : \mathsf{Set} \longrightarrow \mathsf{Pos}$. Hence, $\mathsf{Set} \simeq \mathsf{Pos}^{\mathbb{T}}$ for a finitary monad $\mathbb{T}$. By Example 3.21, the category Set is not exact (in the enriched sense). Hence, Set is not equivalent to *any* variety of ordered algebras by Theorem 5.9. Of course, Set is a quasivariety of ordered algebras, see Example 1.

Remark 5.11. The equivalence of conditions of Theorem 5.9 can be easily extended to the 'many-sorted' case. More in detail: for a category $\mathscr{A}$, the following conditions are equivalent:

1. $\mathscr{A}$ is an S-sorted variety of ordered Σ-algebras for some set S and some λ-ary signature Σ of S-sorted operations.
2. $\mathscr{A}$ is exact and there is an equivalence $\mathscr{A} \simeq [S, \mathsf{Pos}]^{\mathbb{T}}$, for a λ-accessible monad $\mathbb{T}$ on $[S, \mathsf{Pos}]$, where S is a set, considered as a discrete category.
3. $\mathscr{A}$ is exact, has coinserters, and there is a set S and a functor $P : S^{op} \longrightarrow \mathscr{A}$ such that:

   a. Colimits $X * P$ exist for every functor $X : S \longrightarrow \mathsf{Pos}$.

   b. P is a λ-presentable object in $[S, \mathsf{Pos}]$.

   c. P is projective w.r.t. so-morphisms in $[S, \mathsf{Pos}]$.

   d. P is an so-generator, i.e. the canonical $\varepsilon_A : \mathscr{A}(P-, A) * P \longrightarrow A$ is an so-morphism.

Above, by a many-sorted variety we mean the following: given a fixed set S of sorts, we define an *S-sorted λ-ary signature* Σ to consist of operation symbols $\sigma : (s_i \mid i < \lambda) \longrightarrow s$. An S-sorted algebra for Σ consists of an object $X = (X_s \mid s \in S)$ of $[S, \mathsf{Pos}]$ together with a monotone map $[\![\sigma]\!]_X : \prod_{i<\lambda} X_{s_i} \longrightarrow X_s$ for every operation symbol $\sigma : (s_i \mid i < \lambda) \longrightarrow s$ in Σ. Homomorphisms between Σ-algebras are defined in the expected way: they are the morphisms $(f_s \mid s \in S) : (X_s \mid s \in S) \longrightarrow (Y_s \mid s \in S)$ in $[S, \mathsf{Pos}]$ that preserve the operations specified by Σ. The description of the resulting category $\mathsf{Alg}(H_\Sigma)$ of algebras and homomorphisms by means of the corresponding polynomial functor $H_\Sigma : [S, \mathsf{Pos}] \longrightarrow [S, \mathsf{Pos}]$ in the manner of Definition 5.2 can be made. We refer to Kelly and Power (1993) for details.

Given an S-sorted signature Σ, an *S-sorted variety* consists of algebras satisfying inequalities of the form

$$t'(x_k) \sqsubseteq t(x_k)$$

where t' and t are S-sorted terms of the same sort.

Example 5.10 exhibited a finitary monad $\mathbb{T}$ on the category Pos such that $\mathsf{Pos}^{\mathbb{T}}$ is not a variety of ordered algebras. Next example shows that a category of the form $\mathsf{Pos}^{\mathbb{T}}$, $\mathbb{T}$ a finitary monad, need not even be a quasivariety of ordered algebras (on the other hand there are also quasivarieties that are not monadic).

Example 5.12 (Category of the form $\mathsf{Pos}^{\mathbb{T}}$ that is not a quasivariety). Let $\mathbb{T}$ be the monad of the adjunction $F \dashv U : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ with $UX = [\mathbb{2}, X]$ and $FX = X \bullet \mathbb{2}$. The adjunction $F \dashv U$ is *not monadic*, since $\mathbb{2}$ is not projective w.r.t. so-morphisms. This result is in contrast with the case of ordinary categories. See, e.g. Métayer (2004) for discussion of monadicity of functors of the form $[S, -] : \mathscr{C} \longrightarrow \mathscr{C}$ in regular ordinary cartesian closed categories $\mathscr{C}$.

Moreover, the monad $\mathbb{T}$ of $F \dashv U$ does not preserve so-morphisms, see Example 4.8. Therefore, the category $\mathsf{Pos}^{\mathbb{T}}$ is not a quasivariety by Bloom and Wright (1983, Section 7, Proposition 2).

The difference between quasivarieties and varieties of ordered algebras is essentially the difference between regularity and exactness, as the next result shows.

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Theorem 5.13 (Intrinsic characterisation of quasivarieties). For $\mathscr{A}$, the following are equivalent:

1. There is a λ-accessible functor $U : \mathscr{A} \longrightarrow \mathsf{Pos}$ such that $(\mathscr{A}, U)$ is a λ-ary quasivariety.
2. $\mathscr{A}$ is regular, has coinserters, and possesses a λ-algebraic generator.

Proof. (1) implies (2). By assumption, there is an adjunction $F \dashv U$. Define P as $F\mathbb{1}$. Then U is necessarily isomorphic to $\mathscr{A}(P, -)$ and F is isomorphic to $- \bullet P$.

We need to prove that $\mathscr{A}$ is regular. Observe first that the counit ε_A of $F \dashv U$ is effective. This follows from the fact that $U\varepsilon_A$ is effective in Pos (being a split epi) and U is assumed to reflect effective morphisms. Hence, Proposition 4.7(4) can be applied: the comparison functor $K : \mathscr{A} \longrightarrow \mathsf{Pos}^{\mathbb{T}}$ is fully faithful. Moreover, $\mathsf{Pos}^{\mathbb{T}}$ is a regular category by Corollary 4.14 and it is a quasivariety by Bloom and Wright (1983, Section 7, Proposition 2).

(R1) $\mathscr{A}$ has finite limits.
 This follows from (Bloom and Wright 1983, Section 4, Corollary 1).

(R2) $\mathscr{A}$ has $(\mathsf{so}, \mathsf{ff})$-factorisations.
 First of all, U preserves and reflects ff-morphisms by Lemma 4.3. Moreover, $\mathscr{A}$ clearly has (effective,ff)-factorisations. Furthermore, K preserves effective morphisms, since U preserves them and $U^{\mathbb{T}}$ reflects them (since $\mathsf{Pos}^{\mathbb{T}}$ is a quasivariety).
 Therefore, K preserves (effective, ff)-factorisations, these being $(\mathsf{so}, \mathsf{ff})$-factorisations in the quasivariety $\mathsf{Pos}^{\mathbb{T}}$. Since K is fully faithful, K reflects so-morphisms and therefore $\mathscr{A}$ has $(\mathsf{so}, \mathsf{ff})$-factorisations.

(R3) so-morphisms are stable under pullbacks.
 This follows from the fact that K is fully faithful, preserves limits, and $\mathsf{Pos}^{\mathbb{T}}$ is regular.

(R4) so-morphisms coincide with the effective morphisms.
 This follows from the above.

We proved that $\mathscr{A}$ is regular. We prove now that P is a λ-algebraic generator.

a. Tensors $X \bullet P$ exist for every poset X.
 This is clear: $X \bullet P \cong FX$.
b. P is a λ-presentable object.
 Clear: $U = \mathscr{A}(P, -)$ is λ-accessible.
c. P is projective w.r.t. so-morphisms.
 Clear: $U = \mathscr{A}(P, -)$ is assumed to preserve so-morphisms.
d. P is an so-generator, i.e. the canonical $\varepsilon_A : \mathscr{A}(P, A) \bullet P \longrightarrow A$ is an so-morphism.
 This was proved already.

(2) implies (1). Let P denote the λ-algebraic generator of $\mathscr{A}$. Define $U = \mathscr{A}(P, -)$. Then U is λ-accessible and Conditions (Q1)–(Q5) for $(\mathscr{A}, U)$ are verified in the same way as in the proof of Theorem 5.9. $\square$

Remark 5.14. Theorems 5.9 and 5.13 above were stated for an abstract category $\mathscr{A}$. Similar results can be stated for a pair $(\mathscr{A}, U)$ consisting of a category $\mathscr{A}$ and a functor $U : \mathscr{A} \longrightarrow \mathsf{Pos}$, since the properties of the algebraic generator P from the above statements reflect the properties of U. More precisely, the algebraic generator P is the representing object of U.

6. Finitary varieties and strongly finitary monads

In case when the signature Σ is *finitary*, i.e. when $\Sigma : |\mathsf{Set}_{fp}| \longrightarrow \mathsf{Pos}$, one can give yet other characterisations of varieties of Σ-algebras.

1. The first characterisation involves the notion of *strongly finitary functors* introduced by Max Kelly and Steve Lack in Kelly and Lack (1993).
   We prove in Theorem 6.9 below that finitary varieties over Pos are precisely the strongly finitary monadic categories over Pos.
2. The notion of strongly finitary functors is closely related to a certain class of weighted colimits, called *sifted*, see e.g. Bourke (2010).
   We prove in Theorem 6.12 that finitary varieties are precisely free cocompletions of their theories under sifted colimits.

Definition 6.1 (Kelly and Lack 1993). A functor $H : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ is *strongly finitary* if it is a left Kan extension of its restriction along the discrete-poset functor $D : \mathsf{Set}_{fp} \longrightarrow \mathsf{Pos}$, where Set_{fp} is the category of finite sets with discrete order on hom-sets.

A monad $\mathbb{T}$ on Pos is *strongly finitary* if its functor is strongly finitary.

Remark 6.2. By definition, a functor $H : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ is strongly finitary iff it has a coend expansion

$$HX = \int^{n:\mathsf{Set}_{fp}} \mathsf{Pos}(Dn, X) \bullet Hn$$

for every poset X.

Since every Dn is a finitely presentable object in Pos, every strongly finitary functor H is *a fortiori* finitary.

Lemma 6.3. Every strongly finitary functor $H : \mathsf{Pos} \longrightarrow \mathsf{Pos}$ preserves so-morphisms.

Proof. Consider an so-morphism $e : A \longrightarrow B$. Then $He : HA \longrightarrow HB$ has a coend expansion

$$\int^{n:\mathsf{Set}_{fp}} \mathsf{Pos}(Dn, e) \bullet Hn : \int^{n:\mathsf{Set}_{fp}} \mathsf{Pos}(Dn, A) \bullet Hn \longrightarrow \int^{n:\mathsf{Set}_{fp}} \mathsf{Pos}(Dn, B) \bullet Hn$$

Since every $\mathsf{Pos}(Dn, e) : \mathsf{Pos}(Dn, A) \longrightarrow \mathsf{Pos}(Dn, B)$ is surjective, so is every

$$\mathsf{Pos}(Dn, e) \bullet Hm : \mathsf{Pos}(Dn, A) \bullet Hm \longrightarrow \mathsf{Pos}(Dn, B) \bullet Hm$$

Thus He is surjective as a colimit of surjections. □

Example 6.4. None of the implications *strongly finitary* $\Rightarrow$ *finitary and* so-*preserving* $\Rightarrow$ *finitary* can be reversed.

1. The functor $T : X \mapsto [2, X \bullet 2]$ is finitary but it does not preserve so-morphisms. This follows from Example 4.8.

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2. Consider the connected-component functor $\pi_0 : \mathsf{Pos} \longrightarrow \mathsf{Pos}$. It preserves so-morphisms and it is finitary. The functor π_0 is, however, not strongly finitary. Suppose it were, then

$$\pi_0(X) = \int^{n:\mathsf{Set}_{fp}} \mathsf{Pos}(Dn, X) \bullet \pi_0 n = \int^{n:\mathsf{Set}_{fp}} \mathsf{Pos}(Dn, X) \bullet n = X$$

   would hold for every poset X: use that $\pi_0 n = n$ for every discrete poset and, for the last equality, use that the inclusion $D : \mathsf{Set}_{fp} \longrightarrow \mathsf{Pos}$ is dense. But $\pi_0(\mathbb{2}) = \mathbb{1} \not\cong \mathbb{2}$.

Incidentally, both T and π_0 have the structure of a monad: for T, consider the adjunction $- \bullet \mathbb{2} \dashv [\mathbb{2}, -]$, and for π_0, consider the (monadic) adjunction $C \dashv U : \mathsf{Set} \longrightarrow \mathsf{Pos}$, where U is the discrete-poset functor and C assigns the set of components to a poset.

It can be proved that $D : \mathsf{Set}_{fp} \longrightarrow \mathsf{Pos}$ exhibits Pos as a *free cocompletion* of Set_{fp} w.r.t. a certain class of colimits that include filtered colimits and an enriched analogue of reflexive coequalizers, namely *quotients of reflexive coherence data* (see below). This follows by a modification of arguments given in Bourke (2010, Section 8.4).

Definition 6.5 (Bourke 2010). Denote by Δ_2 the full simplicial category truncated at stage two. That is, Δ_2 is given by the graph

$$\begin{tikzcd} \Delta_2 \equiv \mathbb{1} \arrow[r, shift left=3, "\delta^1_1"] \arrow[r, shift right=3, "\delta^1_0"'] & \mathbb{2} \arrow[l, "\iota^0_0"'] \arrow[r, shift left=6, "\delta^2_0"] \arrow[r, "\delta^2_1"] \arrow[r, shift right=6, "\delta^2_2"'] & \mathbb{3} \arrow[l, shift right=3, "\iota^1_1"'] \arrow[l, shift left=3, "\iota^1_0"] \end{tikzcd}$$

subject to simplicial equalities. See, e.g. Mac Lane (1971).

A *reflexive coherence datum* in $\mathscr{X}$ is a diagram $R : \Delta_2{}^{op} \longrightarrow \mathscr{X}$. A *quotient* of a reflexive coherence datum $R : \Delta_2{}^{op} \longrightarrow \mathscr{X}$ is a colimit $J * R$, where $J : \Delta_2 \longrightarrow \mathsf{Pos}$ denotes the full inclusion.

Remark 6.6. The category Δ_2^- introduced in Notation 3.10 is a subcategory of Δ_2. Hence, every reflexive coherence datum is a coherence datum (Definition 3.11).

Filtered colimits and quotients of reflexive coherence data form a density presentation in the sense of Kelly (2005) of the fully faithful dense functor $D : \mathsf{Set}_{fp} \longrightarrow \mathsf{Pos}$.

The *saturation* (the *closure*, in the terminology of Albert and Kelly (1988)) of the class of filtered colimits and quotients of reflexive coherence data is the class of weights, called *sifted*. This is in analogy to the case of ordinary sifted colimits introduced by Lair (1996). More in detail: a weight $W : \mathscr{D}^{op} \longrightarrow \mathsf{Pos}$ is called *sifted*, if the n-fold product functor $\Pi_n : [n, \mathsf{Pos}] \longrightarrow \mathsf{Pos}$ preserves W-colimits, for every finite discrete poset n.

Example 6.7. Every filtered colimit and every quotient of a reflexive coherence datum is an example of a sifted colimit. Every reflexive coequaliser is a sifted colimit.

Using various types of sifted colimits, we can give a characterisation of functors preserving sifted colimits. We formulate the result for functors preserving finite limits between exact categories, since this is how we will need it.

Proposition 6.8. Suppose $H : \mathscr{K} \longrightarrow \mathscr{L}$ preserves finite limits and suppose $\mathscr{K}$ and $\mathscr{L}$ are cocomplete exact categories. Then the following are equivalent:

1. H preserves sifted colimits.
2. H preserves filtered colimits and quotients of reflexive coherence data.
3. H preserves filtered colimits and quotients of congruences.

Proof. Clearly, (1) is equivalent to (2). That (2) implies (3) follows from the fact that every congruence is a reflexive coherence datum. For (3) implies (2) it suffices to prove that H preserves quotients of reflexive coherence data. Consider a reflexive coherence datum

$$\mathbb{D} \equiv \begin{tikzcd} X_2 \arrow[r, shift left=4, "d^2_2"] \arrow[r, leftarrow, shift left=2, "i^1_0" description] \arrow[r, "d^2_1" description] \arrow[r, leftarrow, shift right=2, "i^1_1" description] \arrow[r, shift right=4, "d^2_0"'] & X_1 \arrow[r, shift left=2, "d^1_1"] \arrow[r, leftarrow, "i^0_0" description] \arrow[r, shift right=2, "d^1_0"'] & X_0 \end{tikzcd}$$

and observe that, for the quotient $q : X_0 \longrightarrow X$ of $\mathbb{D}$, the congruence $\ker(q)$ has the same cocones as $\mathbb{D}$. □

We can now formulate the first characterisation of finitary varieties.

Theorem 6.9. For a category $\mathscr{A}$, the following conditions are equivalent:

1. $\mathscr{A}$ is equivalent to a variety of algebras for a finitary signature.
2. $\mathscr{A}$ is equivalent to $\mathsf{Pos}^{\mathbb{T}}$ for a strongly finitary monad $\mathbb{T}$ on Pos.

Proof. (1) implies (2). By Theorem 5.9 we know that $\mathscr{A}$ is an exact category and that $\mathscr{A}$ is equivalent to $\mathsf{Pos}^{\mathbb{T}}$ for a finitary monad $\mathbb{T}$ on Pos. Moreover, the monad $\mathbb{T}$ is given by the adjunction $- \bullet P \dashv \mathscr{A}(P, -)$, where P is a free algebra on $\mathbb{1}$.

To prove that the monad $\mathbb{T}$ is strongly finitary, by Proposition 6.8 it therefore suffices to prove that its functor $X \mapsto \mathscr{A}(P, X \bullet P)$ preserves quotients of congruences in Pos. The left adjoint $X \mapsto X \bullet P$ preserves all colimits. And $\mathscr{A}(P, -)$ does preserve quotients of congruences, since $\mathscr{A}$ is a variety.

(2) implies (1). We only need to prove that $\mathsf{Pos}^{\mathbb{T}}$ is an exact category. Since $\mathbb{T}$ preserves so-morphisms by Lemma 6.3, the category $\mathsf{Pos}^{\mathbb{T}}$ is regular by Corollary 4.14. Thus it remains to be proved that congruences are effective in $\mathsf{Pos}^{\mathbb{T}}$. To that end, consider a congruence

$$\sim \equiv \begin{tikzcd} (X_2, a_2) \arrow[r, shift left=2, "d^2_2"] \arrow[r, "d^2_1" description] \arrow[r, shift right=2, "d^2_0"'] & (X_1, a_1) \arrow[r, shift left=2, "d^1_1"] \arrow[r, leftarrow, "i^0_0" description] \arrow[r, shift right=2, "d^1_0"'] & (X_0, a_0) \end{tikzcd}$$

in $\mathsf{Pos}^{\mathbb{T}}$. Then there is $f : X_0 \longrightarrow X$ in Pos such that $U^{\mathbb{T}}(\sim) = \ker(f)$. Since T preserves quotients of congruences, we can form

$$TU^{\mathbb{T}}(\sim) \equiv \begin{tikzcd} TX_2 \arrow[r, shift left=2, "Td^2_2"] \arrow[r, "Td^2_1" description] \arrow[r, shift right=2, "Td^2_0"'] & TX_1 \arrow[r, shift left=2, "Td^1_1"] \arrow[r, leftarrow, "Ti^0_0" description] \arrow[r, shift right=2, "Td^1_0"'] & TX_0 \end{tikzcd}$$

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having $Tf : TX_0 \longrightarrow TX$ as its quotient. Define $a : TX \longrightarrow X$ as the unique mediating map:

$$\begin{tikzcd}
TX_2 \arrow[r, shift left=2, "Td_2^2"] \arrow[r, "Td_1^2" description] \arrow[r, shift right=2, "Td_0^2"'] \arrow[d, "a_2"'] & TX_1 \arrow[d, "a_1"] & TX_0 \arrow[l, "Ti_0^0" description] \arrow[l, shift right=2, "Td_1^1"'] \arrow[l, shift left=2, "Td_0^1"] \arrow[r, "Tf"] \arrow[d, "a_0"] & TX \arrow[d, dotted, "a"] \\
X_2 \arrow[r, shift left=2, "d_2^2"] \arrow[r, "d_1^2" description] \arrow[r, shift right=2, "d_0^2"'] & X_1 & X_0 \arrow[l, "i_0^0" description] \arrow[l, shift right=2, "d_1^1"'] \arrow[l, shift left=2, "d_0^1"] \arrow[r, "f"'] & X
\end{tikzcd}$$

It is then easy to see that (X, a) is a $\mathbb{T}$-algebra and f is a $\mathbb{T}$-algebra homomorphism. Moreover, $\sim = \ker(f)$ in $\mathsf{Pos}^{\mathbb{T}}$. □

We prove now that finitary varieties of ordered algebras are free cocompletions of certain small categories under sifted colimits.

Definition 6.10. Suppose $\mathbb{T} = (T, \eta, \mu)$ is a strongly finitary monad on Pos. By $\mathsf{Th}(\mathbb{T})$ we denote the full subcategory of $\mathsf{Pos}^{\mathbb{T}}$ spanned by free $\mathbb{T}$-algebras on objects of Set_{fp}. The category $\mathsf{Th}(\mathbb{T})$ is called the *theory* of $\mathbb{T}$.

Remark 6.11. The duals of categories of the form $\mathsf{Th}(\mathbb{T})$ are discrete (finitary) Lawvere theories in the sense of Hyland and Power (2006).

The following result states that the category of algebras for $\mathbb{T}$ is the free cocompletion of $\mathsf{Th}(\mathbb{T})$ under sifted colimits. This is the enriched analogue of the classical result. See, e.g. Theorem 4.13 of Adámek *et al.* (2011).

Theorem 6.12. Let $\mathbb{T} = (T, \eta, \mu)$ be a strongly finitary monad on Pos. Then the embedding $E : \mathsf{Th}(\mathbb{T}) \longrightarrow \mathsf{Pos}^{\mathbb{T}}$ exhibits $\mathsf{Pos}^{\mathbb{T}}$ as a free cocompletion of $\mathsf{Th}(\mathbb{T})$ under sifted colimits.

Proof. We will use Proposition 4.2 of Kelly and Schmitt (2005). Since E is fully faithful and $\mathsf{Pos}^{\mathbb{T}}$ cocomplete, we only need to prove that $\mathsf{Pos}^{\mathbb{T}}$ is the closure of $\mathsf{Th}(\mathbb{T})$ under sifted colimits and that every functor $\mathsf{Pos}^{\mathbb{T}}((Tn, \mu_n), -) : \mathsf{Pos}^{\mathbb{T}} \longrightarrow \mathsf{Pos}$, where n is discrete and finite poset, preserves sifted colimits.

1. We prove that every $\mathbb{T}$-algebra is an iterated sifted colimit of $\mathbb{T}$-algebras free on discrete posets. This is done in three steps:
   a. Using quotients of truncated nerves that are reflexive coherence data, one can exhibit every algebra free on a finite poset.
      More in detail: given a finite poset P, exhibit it as a quotient $q : P_0 \longrightarrow P$ of its truncated nerve

$$\mathsf{nerve}(P) \equiv \begin{tikzcd} P_2 \arrow[r, shift left=2, "d_2^2"] \arrow[r, "d_1^2" description] \arrow[r, shift right=2, "d_0^2"'] & P_1 & P_0 \arrow[l, "i_0^0" description] \arrow[l, shift right=2, "d_1^1"'] \arrow[l, shift left=2, "d_0^1"] \end{tikzcd}$$

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in an analogous way as it was done for $\mathbb{2}$ in Example 3.21. Since $\mathsf{nerve}(P)$ can clearly be augmented to form a reflexive coherence datum, we proved that $F^{\mathbb{T}}P$ arises as a sifted colimit of free algebras on finite discrete posets.

b. Further, using filtered colimits, one can exhibit every algebra free on a poset.
More in detail: suppose X is any poset. Then X can be written as a filtered colimit of finite posets. Hence, $F^{\mathbb{T}}X$ is a filtered (hence, sifted) colimit of algebras of the form $F^{\mathbb{T}}P$, where P is a finite poset.

c. Finally, using canonical presentations that are reflexive coequalizers, one can exhibit every $\mathbb{T}$-algebra.
More in detail, given a $\mathbb{T}$-algebra (X, a), consider the diagram

$$(TTX, \mu_{TX}) \underset{\mu_X}{\overset{Ta}{\rightrightarrows}} (TX, \mu_X) \xrightarrow{a} (X, a)$$

that is a reflexive coequaliser in $\mathsf{Pos}^{\mathbb{T}}$. Hence, (TX, a) is a sifted colimit of free algebras.

2. The functor $\mathsf{Pos}^{\mathbb{T}}((Tn, \mu_n), -) \cong \mathsf{Pos}(n, U^{\mathbb{T}}-) = \mathsf{Pos}(n, -) \cdot U^{\mathbb{T}}$, preserves sifted colimits, since every $\mathsf{Pos}(n, -)$ does and $U^{\mathbb{T}}$ preserves filtered colimits and quotients of congruences. Hence, by Proposition 6.8, $U^{\mathbb{T}}$ preserves sifted colimits.

This concludes the proof. □

7. Conclusions and future work

We gave intrinsic characterisations of categories equivalent to (quasi)varieties of ordered algebras in the sense of Stephen Bloom and Jesse Wright. Namely, we showed that, for the notion of an ordered algebra as a poset equipped with monotone operations of discrete arities, such characterisation theorems are very similar to the classical case of unordered algebras (Adámek *et al.* 2011). The only difference to the classical case is the ubiquitous need for the use of 2-dimensional notions. Hence, one can say that ordered universal algebra in the sense of Stephen Bloom and Jesse Wright is the 'poset-version' of the classical set-based universal algebra.

We believe that our work is only an opening study in the direction of understanding ordered universal algebra using categorical methods. In fact, much of the results surveyed in Adámek *et al.* (2011) need to be investigated. Let us mention just a few:

1. The rôle of sifted colimits in the enriched sense in the study of generalised varieties, see Adámek and Rosický (2001) for the classical case. Also, it is not clear how the non-existence of λ-sifted colimits, λ-uncountable, in the set-based case (see Adámek *et al.* (2000)) transfers to the enriched setting.
2. The connection of (quasi)varieties and regular and exact completions of categories enriched over posets. See, e.g. the paper (Vitale 1994) for the ordinary case.
3. The Morita-type theorems concerning Morita equivalence of ordered theories.

Furthermore, is there a categorical universal algebra of algebras with a basis of monotone *and antitone* operations? This could lead to applications of categorical algebra to order-algebraizable logics in the sense of Raftery (2013).

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