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Cotorsion radicals and projective modules

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We study the notion (for categories of modules) dual to that of torsion radical and its connections with projective modules.

Torsion radicals in categories of modules have been studied extensively in connection with quotient categories and rings of quotients. (See [8], [12] and [13].) In this paper we consider the dual notion, which we have called a cotorsion radical. We show that the cotorsion radicals of the category ${}_R M$ correspond to the idempotent ideals of R . Thus they also correspond to TTF classes in the sense of Jans [9].

It is well-known that the trace ideal of a projective module is idempotent. We show that this is in fact a consequence of the natural way in which every projective module determines a cotorsion radical. As an application of these techniques we study a question raised by Endo [7], to characterize rings with the property that every finitely generated, projective and faithful left module is completely faithful. We prove that for a left perfect ring this is equivalent to being an S-ring in the sense of Kasch. This extends the similar result of Morita [15] for artinian rings.

1. Cotorsion radicals

Throughout this paper R will denote an associative ring with identity, and ${}_R M$ and M_R will denote the categories of unital left and right R-modules, respectively. Homomorphisms will be written on the right. The reader is referred to Lambek [11] for basic definitions.

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Following the terminology of Maranda [13], a subfunctor ρ of the identity on an abelian category $\mathcal{A}$ will be called a preradical of $\mathcal{A}$. A preradical ρ is said to be proper if $\rho(A) \neq A$ for some $A \in \mathcal{A}$ and $\rho(A') \neq 0$ for some $A' \in \mathcal{A}$. A preradical ρ of ${}_R M$ is proper if and only if $\rho(R) \neq R$ and $\rho(M) \neq 0$ for some $M \in {}_R M$. A preradical ρ is called idempotent if $\rho^2 = \rho$; it is called a torsion preradical if ρ is a left exact functor; it is called a radical if $\rho(A/\rho(A)) = 0$ for all $A \in \mathcal{A}$. An object $A \in \mathcal{A}$ is ρ-torsion if $\rho(A) = A$ and ρ-torsionfree if $\rho(A) = 0$.

In order to establish notions formally dual to these, we note that a preradical ρ of ${}_R M$ associates with each module ${}_R M$ a short exact sequence

$$0 \to \rho(M) \to M \to M/\rho(M) \to 0$$

and with each R-homomorphism $f : {}_R M \to {}_R N$ a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & \rho(M) & \to & M & \to & M/\rho(M) & \to & 0 \\ & & \downarrow & & \downarrow f & & \downarrow & & \\ 0 & \to & \rho(N) & \to & N & \to & N/\rho(N) & \to & 0 \end{array}$$

where the endomorphisms are those induced by f. (Since ρ is a preradical we must have $(\rho(M))f \subseteq \rho(N)$.) As in [4], we denote by $1/\rho$ the functor which assigns to each module ${}_R M$ the module $M/\rho(M)$ and to each R-homomorphism f the homomorphism induced by f in the diagram above.

If we use $*$ to denote the dual of $1/\rho$ in the dual category ${}_R M^*$ of ${}_R M$, then $(1/\rho)^*$ is a preradical, and since ρ is a radical if and only if $(1/\rho)^2 = 1/\rho$, it follows that ρ is a radical if and only if $(1/\rho)^*$ is an idempotent preradical. Furthermore, ρ is an idempotent preradical if and only if $(1/\rho)^*$ is a radical, so these two notions are dual.

PROPOSITION 1.1. *Let* ρ *be a preradical of* ${}_R M$. *The following conditions are equivalent:*

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(i) $(1/\rho)^*$ *is a torsion radical of* ${}_R M^*$ *;*

(ii) ρ *is idempotent and* $1/\rho$ *is right exact;*

(iii) ρ *is idempotent and every epimorphism* $\pi : {}_R M \to {}_R N$ *induces an epimorphism* $\rho(\pi) : \rho(M) \to \rho(N)$.

Proof. Conditions *(i)* and *(ii)* are easily seen to be equivalent using the definitions and remarks preceding the proposition. To show the equivalence of *(ii)* and *(iii)*, given an exact sequence $0 \to M' \to M \to M'' \to 0$ in ${}_R M$, consider the induced diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & \rho(M') & \to & \rho(M) & \to & \rho(M'') & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & M' & \to & M & \to & M'' & \to 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & M'/\rho(M') & \to & M/\rho(M) & \to & M''/\rho(M'') & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & & 0 & & 0 &
\end{array}$$

In the diagram the columns are exact and $M/\rho(M) \to M''/\rho(M'')$ is an epimorphism. A standard diagram chasing argument shows that

$$M'/\rho(M') \to M/\rho(M) \to M''/\rho(M'')$$

is exact at $M/\rho(M)$ if and only if $\rho(M) \to \rho(M'')$ is an epimorphism. This completes the proof of the proposition. //

It is clear that a preradical which satisfies condition *(iii)* of the above proposition must be a radical, and so we make the following definition.

DEFINITION 1.2. A preradical ρ in ${}_R M$ will be called a cotorsion radical if ρ is idempotent and every epimorphism $\pi : {}_R M \to {}_R N$ induces an epimorphism $\rho(\pi) : \rho(M) \to \rho(N)$.

PROPOSITION 1.3. *Let* ρ *be a preradical of* ${}_R M$. *The following conditions are equivalent:*

(i) every epimorphism $\pi : {}_R M \to {}_R N$ *induces an epimorphism*

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$\rho(\pi) : \rho(M) \to \rho(N)$;

(ii) $\rho(M) = \rho(R)\cdot M$ *for all* $M \in {}_R\mathsf{M}$;

(iii) ρ *is a radical and every factor module of a* ρ*-torsionfree module is* ρ*-torsionfree.*

Proof. *(i)* ⇒ *(ii)*. Let $M \in {}_R\mathsf{M}$ and let R^I be the direct sum of I copies of R, where the index set I is just M itself. Then the homomorphisms $f_m : R \to M$ defined for all $m \in M$ by $(r)f_m = rm$ together define an epimorphism $f : R^I \to M$. Since $\rho(R^I)$ consists precisely of those elements of R^I for which each component belongs to the ideal $\rho(R)$, it follows on assuming condition *(i)* that

$$\rho(M) = \left(\rho\left(R^I\right)\right)f = \rho(R)\cdot M .$$

(ii) ⇒ *(iii)*. For all $M \in {}_R\mathsf{M}$, $\rho(R)\cdot\left(M/\rho(R)\cdot M\right) = 0$, and if $\rho(R)\cdot M = 0$, then $\rho(R)\cdot M'' = 0$ for every factor module M'' of M.

(iii) ⇒ *(i)*. Let $\pi : {}_RM \to {}_RN$ be an epimorphism. Then π induces an epimorphism $M/\rho(M) \to N/\left(\rho(M)\right)\pi$, and so if condition *(iii)* is satisfied, $M/\rho(M)$ is ρ-torsionfree and $N/\left(\rho(M)\right)\pi$ must be ρ-torsionfree as well. It follows that we must have $\rho(N) \subseteq \left(\rho(M)\right)\pi$, and therefore $\rho(N) = \left(\rho(M)\right)\pi$. //

The above proposition generalizes a result of Kurata [*10*, Lemma 2.1]. Moreover, if A is any ideal of R, the proof shows that setting $\rho(M) = A\cdot M$ for all $M \in {}_R\mathsf{M}$ defines a preradical of ${}_R\mathsf{M}$ which satisfies the conditions of Proposition 1.3. It follows immediately that there is a one-to-one correspondence between ideals of R and radicals of ${}_R\mathsf{M}$ satisfying the conditions of Proposition 1.3. (Note that for such a radical ρ the ρ-torsionfree R-modules are nothing more than the left $R/\rho(R)$-modules.) In this situation the radical ρ is idempotent if and only if $\rho(R)$ is an idempotent ideal, so we have proved the following theorem.

THEOREM 1.4. *There is a one-to-one correspondence between cotorsion*

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radicals of ${}_R\mathrm{M}$ *and idempotent ideals of* R .

A class T of modules is called by Jans [9] a TTF class if it is closed under taking submodules, factor modules, direct products, and if in any exact sequence $0 \to M' \to M \to M'' \to 0$, $M', M'' \in T$ implies $M \in T$. A TTF class is the torsionfree class of a cotorsion radical, and conversely the torsionfree class of a cotorsion radical is a TTF class, so Theorem 1.4 gives another proof of Corollary 2.2 of Jans [9], that TTF classes correspond to idempotent ideals.

If $M, N \in {}_R\mathrm{M}$, we denote by $\mathrm{tr}_N(M)$ the sum in N of all R-homomorphic images of M . The functor rad^M defined by $\mathrm{rad}^M(N) = \mathrm{tr}_N(M)$, for all $N \in {}_R\mathrm{M}$, is an idempotent preradical. The following proposition gives a new way of viewing some known results concerning the trace ideal of a projective module. (See the appendix in the paper by Auslander and Goldman [1].)

PROPOSITION 1.5. *If* ${}_RP$ *is projective, then* rad^P *is a cotorsion radical.*

Proof. For $N \in {}_R\mathrm{M}$, $\mathrm{rad}^P(N) = 0 \iff \mathrm{hom}_R(P, N) = 0$. If P is projective, then it follows that $\mathrm{hom}_R\big(P, N/\mathrm{rad}^P(N)\big) = 0$ for all $N \in {}_R\mathrm{M}$, since any non-zero homomorphism $f : P \to N/\mathrm{rad}^P(N)$ lifts to a homomorphism $g : P \to N$ with $(P)g \not\subseteq \mathrm{rad}^P(N)$, a contradiction. Furthermore, if $\mathrm{hom}_R(P, N) = 0$, then $\mathrm{hom}_R(P, N'') = 0$ for every factor module N'' of N . We may conclude from Proposition 1.3 *(iii)* that ρ is a cotorsion radical. //

A partial converse to Proposition 1.5 can be obtained. A module ${}_RM$ is called torsionless if it can be embedded in a direct product of copies of ${}_RR$. Any projective module is torsionless, and any submodule of a torsionless module is torsionless.

PROPOSITION 1.6. *If* ρ *is a cotorsion radical of* ${}_R\mathrm{M}$ *, then*

$\rho = \text{rad}^M$ *for a torsionless module* ${}_RM$.

Proof. If ρ is a cotorsion radical of ${}_R\mathsf{M}$, let ${}_RM = \rho(R)$. As a left ideal, $\rho(R)$ is torsionless, and since ρ is idempotent, it is ρ-torsion. Therefore we must have $(M)f \subseteq \rho(N)$, for all $N \in {}_R\mathsf{M}$ and $f \in \text{hom}_R(M, N)$. Thus $\text{rad}^M(N) \subseteq \rho(N)$. On the other hand, $\rho(N) = \rho(R)\cdot N \subseteq \text{tr}_N(\rho(R)) = \text{rad}^M(N)$, for all $N \in {}_R\mathsf{M}$. //

If $\pi : {}_RP \to {}_RM$ is an epimorphism, with P projective, then P is called a projective cover of M if $\ker(\pi)$ is small in P , that is, if for all submodules P' of P , $P = P' + \ker(\pi)$ implies $P' = P$. The notion of projective cover was defined and studied by Bass [3]. He called a ring R left perfect if every left R-module has a projective cover.

LEMMA 1.7. *Let* ${}_RM$ *be a module with a projective cover* $\pi : {}_RP \to {}_RM$. *If* ρ *is a cotorsion radical of* ${}_R\mathsf{M}$ *and* M *is* ρ-*torsion, then* P *is* ρ-*torsion.*

Proof. The proof depends only on the fact that $\ker(\pi)$ is small. If M is ρ-torsion, then $\rho(M) = M$, and since ρ is a cotorsion radical, we must have $(\rho(P))\pi = \rho(M) = M$. Thus $\rho(P) + \ker(\pi) = P$, and since $\ker(\pi)$ is small, this implies that $\rho(P) = P$. //

THEOREM 1.8. *If every idempotent ideal of* R *has a projective cover as a left* R-*module, then every cotorsion radical of* ${}_R\mathsf{M}$ *is of the form* rad^P *for a projective module* ${}_RP$.

Proof. Assume the given condition and let ρ be a cotorsion radical, $M = \rho(R)$, and $P \to M$ be a projective cover of M . By Proposition 1.6, $\rho = \text{rad}^M$, and since M is a homomorphic image of P it follows that $\rho(N) = \text{rad}^M(N) \subseteq \text{rad}^P(N)$, for all $N \in {}_R\mathsf{M}$. By Lemma 1.7, P is ρ-torsion, and therefore $\text{rad}^P(N) \subseteq \rho(N)$, for all $N \in {}_R\mathsf{M}$. //

We note that the condition of Theorem 1.8 is satisfied in any of the following cases:

(i) R is left perfect;

(ii) R is left semi-perfect and left noetherian;

(iii) R is left hereditary.

(Recall that a ring R is left semi-perfect if every finitely generated left R-module has a projective cover, and left hereditary if every left ideal is projective.)

2. Faithful preradicals

DEFINITION 2.1. A preradical ρ of ${}_R\mathrm{M}$ will be called faithful if $\rho(M)$ is faithful for every faithful module ${}_RM$.

Recall that ${}_RM$ is faithful if the annihilator ideal $\mathrm{ann}(M) = \{r \in R \mid rM = 0\}$ is the zero ideal. For a left ideal A of R we write $l(A)$ rather than $\mathrm{ann}(A)$.

PROPOSITION 2.2. *Let* ρ *be a preradical of* ${}_R\mathrm{M}$. *Then* ρ *is faithful if and only if* $l(\rho(R)) = 0$.

Proof. ONLY IF: This is obvious from the definition.

IF: If $l(\rho(R)) = 0$ and ${}_RM$ is faithful, then $\rho(R)\cdot M$ is faithful. Since $\rho(M) \supseteq \rho(R)\cdot M$ it is clear that $\rho(M)$ is faithful. //

The ring R is prime if for any ideals A, B of R, $A\cdot B = 0$ implies $A = 0$ or $B = 0$. It follows from Proposition 2.2 that if R is prime then a preradical ρ is faithful if and only if $\rho(R) \neq 0$. In fact, this property characterizes prime rings.

PROPOSITION 2.3. *Let* ${}_RM \in {}_R\mathrm{M}$. *Then* rad^M *is faithful if and only if* M *has a faithful, torsionless factor module.*

Proof. ONLY IF: If rad^M is faithful, let M' be the intersection of all kernels of homomorphisms from ${}_RM$ to ${}_RR$, and $M'' = M/M'$. Then M'' is torsionless and $\mathrm{tr}_R(M) = \mathrm{tr}_R(M'')$. Since rad^M is faithful we have $l(\mathrm{tr}_R(M'')) = 0$, and a short argument can be given to show that M'' must therefore be faithful.

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IF: Suppose that M has a faithful, torsionless factor module M''. Then $\text{tr}_R(M'') \subseteq \text{tr}_R(M)$, so it suffices to show that $l(\text{tr}_R(M'')) = 0$. Since M'' is faithful, given $0 \neq r \in R$ there exists $m \in M''$ with $rm \neq 0$, and then since M'' is torsionless there exists $f \in \text{hom}_R(M'', R)$ such that $(rm)f \neq 0$. But then $r((m)f) \neq 0$, and $(m)f \in \text{tr}_R(M'')$. //

Azumaya [2] calls a module ${}_RM$ completely faithful if $\text{tr}_R(M) = R$. The next proposition is related to §3 on faithful projective modules, but it is stated here in terms of cotorsion radicals. It follows from the correspondence between cotorsion radicals and idempotent ideals that ${}_RM$ has no proper faithful cotorsion radicals if and only if $l(A) \neq 0$ for every proper idempotent ideal A of R. If, moreover, every cotorsion radical is of the form rad^P for a projective module ${}_RP$, then by Proposition 2.2 and Proposition 2.3 this condition is equivalent to the condition that every faithful, projective left R-module is completely faithful. A ring R is called a left S-ring if $l(A) \neq 0$ for every proper right ideal A of R.

PROPOSITION 2.4. *Let R be a left perfect ring. Then ${}_RM$ has no proper faithful cotorsion radicals if and only if R is a left S-ring.*

Proof. IF: If R is a left S-ring, then certainly $l(A) \neq 0$ for every proper idempotent ideal A, and the result follows from Theorem 1.4.

ONLY IF: Lambek [11, Chapter 4] calls a right ideal D of R dense if for all $f \in \text{hom}_R(R, E(R_R))$, $f = 0$ if $(D)f = 0$, where $E(R_R)$ is the injective envelope of the right R-module R_R. Equivalently, D is dense if and only if $\text{hom}_R(R/D, E(R_R)) = 0$.

Jans, in Theorems 2.1 and 3.1 of [9], has shown that if R is left perfect, then the intersection D_0 of all dense right ideals is a dense, idempotent ideal. Lambek has shown [11, Corollary, p. 96] that an ideal D is dense as a right ideal if and only if $l(D) = 0$. Thus we must have $l(D_0) = 0$. Then from our assumption it follows that $D_0 = R$, and by Theorem 3.2 of Jans [9], R is a left S-ring. //

Our proof depended only on the fact that the intersection of dense right ideals is again dense, which, as shown by Jans, holds for a larger class of rings than just the class of left perfect rings. Alin and Armendariz [0] and Dlab [5] have given proofs that in a left perfect ring the intersection of right ideals dense with respect to any torsion radical of M_R is again dense. (A right ideal D is dense with respect to a torsion radical ρ of M_R if $\rho(R/D) = R/D$.)

3. Faithful projective modules

Azumaya in [2] characterized rings for which every faithful left R-module is completely faithful. Endo [7, §6] investigated conditions under which every finitely generated, projective and faithful left R-module is completely faithful. We show in this section that if R is left perfect then this condition is satisfied if and only if R is a left S-ring. This generalizes a result of Morita [15, Theorem 1]. We first give some general results.

If ${}_RP$ is finitely generated and projective, then for some positive integer n there exists an epimorphism $\pi : R^n \to P$, and this splits by a monomorphism $\theta : P \to R^n$. Since $\pi\theta$ is an endomorphism of a free module, it can be described by an $n \times n$ matrix X with entries in R , and X is idempotent. Let $\langle X \rangle$ denote the (two-sided) ideal generated by the entries of the matrix X . Although the matrix X is not uniquely determined by P , the next lemma shows that the ideal $\langle X \rangle$ is uniquely determined.

LEMMA 3.1. *Let* ${}_RP$ *be finitely generated and projective, and* X *an associated idempotent matrix. Then* $\langle X \rangle = \mathrm{tr}_R(P)$.

Proof. We first show that $\langle X \rangle \subseteq \mathrm{tr}_R(P)$. Since $\mathrm{tr}_R(P)$ is an ideal, we only need to show that the generators of $\langle X \rangle$ are contained in $\mathrm{tr}_R(P)$. This follows immediately upon considering the projections $(R^n)X \to R^n \to R$, where $X : R^n \to P \to R^n$.

To show that $\mathrm{tr}_R(P) \subseteq \langle X \rangle$, it suffices to show that $(P)f \subseteq \langle X \rangle$ for all $f \in \hom_R(P, R)$. Any such R-homomorphism f can be extended to $g : R^n \to R$, and then the components of this extension are determined by multiplication on the right by elements of R . Thus $(P)f = (R^n)Xg$, and so $(P)f \subseteq \langle X \rangle$. //

The above lemma gives one method of characterizing trace ideals of finitely generated, projective modules. It can be extended to a characterization of trace ideals of arbitrary projective modules by using row-finite matrices. Since a projective module ${}_RP$ is faithful if and only if $l(\mathrm{tr}_R(P)) = 0$ (Propositions 2.2 and 2.3), we have the following proposition.

PROPOSITION 3.2. *Every finitely generated, projective and faithful left R-module is completely faithful if and only if* $l(A) \neq 0$ *for every proper idempotent ideal* A *of* R *which is generated as a two-sided ideal by the entries of a finite idempotent matrix over* R .

Although this characterization is somewhat unwieldy, it can be used to give a short proof of the well-known fact that if R is a commutative ring, then every finitely generated, projective and faithful R-module is completely faithful. To show this, suppose that X is a finite idempotent matrix with entries in a commutative ring R . Let E be the identity matrix and let $\widetilde{E - X}$ be the adjoint matrix of $E - X$. Since X is idempotent, $(E-X)(X) = (0)$, and therefore $(\det(E-X))(X) = (\widetilde{E-X})(E-X)(X) = (0)$. If $l(\langle X \rangle) = 0$, then this implies that $\det(E-X) = 0$, which shows that $1 \in \langle X \rangle$ and $\langle X \rangle = R$. Thus, in a commutative ring, any proper ideal generated by a finite idempotent matrix must have non-zero annihilator.

The next theorem is the main result of the section.

THEOREM 3.3. *Let* R *be a left perfect ring. Then the following conditions are equivalent:*

(i) R *is a left S-ring;*

(ii) $l(A) \neq 0$ *for every proper idempotent ideal* A *of* R *;*

(iii) every projective and faithful left R-module is completely

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faithful;

(iv) *every finitely generated, projective and faithful left R-module is completely faithful;*

(v) $l(A) \neq 0$ *for every proper idempotent ideal* A *of* R *of the form* $\langle e \rangle$, *for an idempotent element* $e \in R$.

Proof. From previous results, it follows that (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) for any ring R . (That (iv) ⇒ (v) follows from Proposition 3.2.)

Michler [14, Proposition 2.1] has shown that if R is left perfect, then every idempotent ideal of R is of the form $\langle e \rangle$ for some idempotent element e of R . Thus (v) ⇒ (ii) if R is left perfect. Again if we assume that R is left perfect, then (ii) ⇒ (i) by applying Proposition 2.4 and the correspondence between proper cotorsion radicals of ${}_R M$ and proper idempotent ideals of R . This completes the proof. //

Certain of the conditions in the above theorem are equivalent in more general circumstances. For instance, (ii) and (iii) are equivalent if R is left hereditary. If R is left hereditary and left noetherian, then conditions (ii) - (iv) are equivalent, a result of Endo [7, Corollary 6.3]. Conditions (i) - (ii) are equivalent whenever the intersection of dense right ideals is dense.

We conclude with an observation giving some conditions under which every non-zero projective module is completely faithful. Recall that a left and right hereditary, left and right noetherian prime ring which is a maximal order in its quotient ring is said to be a Dedekind prime ring.

PROPOSITION 3.4. *Let* R *be hereditary and noetherian. The following conditions are equivalent:*

(i) ${}_R M$ *has no proper cotorsion radicals;*

(ii) *every non-zero projective* R*-module is completely faithful;*

(iii) R *is a Dedekind prime ring.*

Proof. (i) ⟺ (ii). Conditions (i) and (ii) are both equivalent to the condition that R has no proper idempotent ideals, since R is left and right hereditary.

(ii) $\Leftrightarrow$ *(iii)*. Each non-zero ideal A of R is projective, so if condition *(ii)* holds, then every ideal A is completely faithful, and thus $l(A) \neq 0$. This shows that assuming *(ii)* implies that R is a prime ring. The result then follows from Theorem 1.2 of Eisenbud and Robson [6] which states that an hereditary, noetherian prime ring R is a Dedekind prime ring if and only if R has no proper idempotent ideals. //

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