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SEQUENTIALLY RELATIVELY UNIFORMLY COMPLETE RIESZ SPACES AND VULIKH ALGEBRAS

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Throughout this paper V will denote an Archimedean Riesz space with a weak unit e and a zero element θ. A sequence $f_1, f_2, f_3, \ldots$ of points of V is said to converge *relatively uniformly* to a point f (with regulator the point g of V) if, for each $\epsilon > 0$, there is a number N such that, if n is a positive integer and $n > N$, then $|f - f_n| < \epsilon g$. In an Archimedean Riesz space a relatively uniformly convergent sequence has a unique limit. The sequence $f_1, f_2, f_3, \ldots$ is called a *relatively uniform Cauchy sequence* (with regulator g) if, for each $\epsilon > 0$, there is a number N such that if n and m are positive integers and $n, m > N$, then $|f_n - f_m| < \epsilon g$. A subset M of V is said to be *sequentially relatively uniformly complete*, written s.r.u.-complete, whenever every relatively uniform Cauchy sequence of points of M (with regulator in V) converges to a point of M. This property was defined by Luxemburg and Moore in [**4**] and some related conditions were derived. The property of being Archimedean and s.r.u.-complete is intermediate to the properties of being Archimedean and being σ-complete (see Vulikh [**7,** p. 127]). Several important Riesz spaces, such as C[0, 1], QC[0, 1] (the space of all quasi-continuous functions on the interval [0, 1]), $\mathrm{B}_\alpha[0, 1]$ (the αth Baire class, α finite), and the space of all functions on [0, 1] which are Rieman-Stieltjes integrable with respect to a given function of bounded variation are Archimedean and s.r.u.-complete, but not σ-complete.

The pair (V, e) will be said to be a *Vulikh algebra* if a multiplication can be defined on V which makes it an associative, commutative algebra with multiplicative unit e which is positive in the sense that if $f \geqq \theta$ and $g \geqq \theta$ then $fg \geqq \theta$. For some properties of Vulikh algebras see Rice [**5**] or Vulikh [**7**].

When necessary, it will be assumed that V is a subspace of the set of all almost finite extended real valued continuous functions on an extremally disconnected compact Hausdorff space S and that e is the function identically equal to 1. If each of f and g belong to V their pointwise product will be defined as follows: Both f and g are finite on a dense subset Q of S. Their pointwise product on Q is a continuous function on Q and can be extended uniquely to a continuous function on S, since S is extremally disconnected.

There is at most one multiplication which makes (V, e) a Vulikh algebra (see [**3**] or [**1**, Theorem 5.1]). If (V, e) is a Vulikh algebra and it is represented as a Riesz space as a subspace of the set of all almost finite extended rea

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valued continuous functions on a extremally disconnected compact Hausdorff space with e the constant function 1, then the Vulikh algebra multiplication will be the same as the pointwise multiplication described above.

THEOREM 1. *Suppose V is* s.r.u.-*complete and e is a strong unit. Then (V, e) is a Vulikh algebra.*

In this case, V can be represented as the Riesz space C of all real-valued continuous functions on a compact Hausdorff space (see [**2**, Theorem 4.1]). Consequently, V is a Vulikh algebra since C is.

THEOREM 2. *Suppose V is* s.r.u.-*complete. There is an ideal M of V such that (M, e) is the largest Vulikh algebra contained in V, i.e., if H is a sub-Riesz space of V such that (H, e) is a Vulikh algebra then H is a subset of M.*

Proof. Let γ be the collection to which the sub-Riesz space K of V belongs if and only if K contains e and (K, e) is a Vulikh algebra. Let N be the ideal generated by e. By Theorem 1, (N, e) is a Vulikh algebra and therefore N belongs to γ. Order γ by inclusion. It follows from the remark about the uniqueness of multiplication in a Vulikh algebra that if K and J are in γ and J is a subset of K then the multiplication on K restricted to J agrees with the multiplication on J. By a Zorn's lemma argument there exists a maximal set M of γ containing N.

Suppose H is a set in γ. We wish to show that H is a subset of M. If it can be shown that for each point f of H and g of M that fg is in V, then $H \cup M$ generates a Vulikh algebra which would contradict the maximality of M unless H is a subset of M.

(1) Suppose f is a positive point of H and g is a positive point of N. It can be assumed that $g \leqq e$. If n is a positive integer, then $f \leqq (1/n)f^2 + ne$, and also $ne \leqq (1/n)f^2 + ne$, so that $f \vee ne - ne \leqq (1/n)f^2$. Therefore,

$$\begin{aligned}\theta \leqq fg - (f \wedge ne)g &= fg - (f + ne - f \vee ne)g \\ &= (f \vee ne - ne)g \leqq f \vee ne - ne \leqq (1/n)f^2.\end{aligned}$$

As $f \wedge ne$ is in N, $(f \wedge ne)g$ is in N, and fg is in V as V is s.r.u.-complete.

(2) Suppose f is a positive point of H and g is a positive point of M. Then $\theta \leqq fg - f(g \wedge ne) = f(g - g \wedge ne) = f(g \vee ne - ne)$. We wish to show that $f(g \vee ne - ne) \leqq (1/n)(f^3 \vee g^3)$. Suppose x is in S. If $g(x) \leqq n$, then $(g \vee ne - ne)(x) = 0$ and $f(g \vee ne - ne)(x) = 0 \leqq (1/n)(f^3 \vee g^3)(x)$. If $g(x) \geqq n$, then either $f(x) \geqq g(x)$ or $g(x) \geqq f(x)$. If $f(x) \geqq g(x)$ then, $f(g \vee ne - ne)(x) \leqq f^2(x) \leqq (1/n)f^3(x) \leqq (1/n)(f^3 \vee g^3)(x)$. If $g(x) \geqq f(x)$, then $f(g \vee ne - ne)(x) \leqq g^2(x) \leqq (1/n)g^3(x) \leqq (1/n)(g^3 \vee f^3)(x)$. Thus fg is in V, since $f(g \wedge ne)$ is in V for all n by (1).

As H and M are lattice ordered (2) is sufficient to show that for any point f of H and any point g of M, fg is in V. Therefore H is a subset of M.

Now suppose each of f and g is a positive point of M and h is a point of V such that $\theta \leqq h \leqq f$. The sequence $\{g(h \wedge ne)\}$ converges relatively uniformly to gh with regulator $f^3 \vee g^3$. Thus gh is in V. Similarly if h^p is in V for some positive integer p, as $\theta \leqq h^p \leqq f^p$, $h^p g$ is in V for each point g of M. To show that h is in M it is sufficient to show that h^p is in V for each positive integer p, because, then $\{h\} \cup M$ would generate a Vulikh algebra, which would contradict the maximality of M unless h belongs to M.

Suppose p is a positive integer such that h^p is in V. Then $\theta \leqq h^{p+1} - h^p(h \wedge ne) = h^p(h - h \wedge ne) \leqq (1/n)f^{p+2}$ and h^{p+1} is in V. Thus, by induction h^p is in V for each positive integer p, h is in M, and M is an ideal.

The following is a generalization of Theorem 1. (Note that in the following M is assumed to contain the limit of a relative uniform Cauchy sequence where the regulator may be in V, not just in M.)

THEOREM 3. *Suppose (V, e) is a Vulikh algebra and M is a* s.r.u.*-complete sub-Riesz space of V containing e. Then (M, e) is a Vulikh algebra.*

Proof. Suppose each of f and g is a positive point of M. Let N be the ideal of M generated by e. By Theorem 1, (N, e) is a Vulikh algebra. If $g \leqq e$, $(f \wedge ne)g$ is in M. Hence

$$\theta \leqq fg - (f \wedge ne)g = fg - (f + ne - f \vee ne)g$$
$$= (f \vee ne - ne)g \leqq f \vee ne - ne \leqq (1/n)f^2$$

and fg belongs to M. If it is not assumed that either f or g is in N, then $f(g \wedge ne)$, $n = 1, 2, 3, \ldots$, is a sequence of points of M converging relatively uniformly to fg with regulator $f^4 \vee g^4$.

The following theorem gives a necessary and sufficient condition for (V, e) to be a Vulikh algebra under the assumption that V is s.r.u.-complete. Three sufficient conditions for (V, e) to be a Vulikh algebra were known before. One was that V be σ-complete and have a strong unit [**7**]. This is generalized by Theorem 3 of this paper. Another was that V be complete and that every pairwise disjoint subset of the positive cone of V have a supremum (see [**5**] or [**6**]). Neither of these conditions are necessary. The condition given here is much weaker than either of these. Conrad and Diem [**1**, Theorem 5.1] give a necessary and sufficient condition that (V, e) be a Vulikh algebra with no further assumptions than that V is an Archimedean Riesz space and e is a weak unit. The following condition appears to be different in nature from theirs.

THEOREM 4. *Suppose V is* s.r.u.*-complete. Then (V, e) is a Vulikh algebra if and only if for each point $f \geqq \theta$ of V there is a point g of V such that $(1/n)g \geqq |f - f \wedge ne|$, $n = 1, 2, 3, \ldots$.*

Proof. If (V, e) is a Vulikh algebra then f^2 has the property required of g.

Suppose that for each positive f such a g exists. Then

$$(1/n)g \geqq f - f \wedge ne,$$
$$(1/n)g \geqq f \vee ne - ne,$$
$$(1/n)g \geqq f - ne, \text{ and}$$
$$g \geqq nf - n^2e.$$

Suppose that $f \geqq 5e$. Then $f(x) = k \cdot n$ where n is a positive integer and $3/2 \leqq k \leqq 2$. So $g(x) \geqq nkn - n^2 = (k-1)n^2 = (k-1)(f(x)/k)^2 = ((k-1)/k^2)f^2(x) \geqq (2/9)f^2(x)$. If $f \ngeqq 5e$, there exists an element g of V such that $g \geqq (2/9)(f \vee 5e)^2 \geqq (2/9)f^2$. Thus for each $f \geqq \theta$, there is a point d of V such that $d \geqq f^2$. Since d is a positive point of V, the same process can be applied to d and hence there is a point r of V such that $r \geqq d^2 \geqq f^4$.

Then suppose each of h and k is a positive point of V. Let s be a point of V such that $s \geqq h^4$ and let t be a point of V such that $t \geqq k^4$. By Theorem 1, $(h \wedge ne)(k \wedge pe)$ belongs to V. The sequence $(h \wedge ne)(k \wedge pe), p = 1, 2, \ldots,$ converges relatively uniformly to $(h \wedge ne)k$ with regulator $s \vee t$. The sequence $(h \wedge ne)k$, $n = 1, 2, \ldots,$ converges relatively uniformly to hk with regulator $s \vee t$.

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