# TOPOLOGIES OF LATTICE PRODUCTS

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**1. Introduction.** A number of different ways of defining topologies in a lattice or partially ordered set in terms of the order relation are known. Three of these methods have proved to be useful and convenient for lattices of special types, namely the *ideal* topology, the *interval* topology, and the *new interval* topology of Garrett Birkhoff. In another paper **(2)** we have shown that these three topologies are equivalent for chains (totally ordered sets), where they reduce to the usual intrinsic topology of the chain.

Since many important lattices are either direct products of chains or sublattices of such products, it is natural to ask what relationships exist between the various order topologies of a direct product of lattices and those of the lattices themselves. In particular it would be useful to know when an order topology of a lattice product is equivalent to the cartesian product of the order topologies of the factors. We answer some of these questions in this paper.

We show that the ideal topology of a finite product of lattices is equivalent to the cartesian product of the ideal topologies. For an infinite product the topologies may be distinct. We prove that the new interval topology of a finite product of chains is equivalent to the cartesian topology, but that for an infinite product of chains the two topologies may differ.

We show that the interval topology of a finite or infinite product of bounded lattices is equivalent to the cartesian topology. For products of lattices without extreme elements the interval topology may be unsatisfactory. However, for the case of a finite product of chains, each having a smallest element (or a largest element), we show that the interval topology is equivalent to the cartesian topology. We study also the relationship between the three topologies, and show that for finite products of chains, the ideal and new interval topologies are equivalent.

**2. The ideal topology of products.** The ideal topology of a lattice is obtained by taking the completely irreducible ideals and dual ideals as a subbase for the open sets **(9)**. An ideal $I$ of a lattice is a subset such that $a \cup b \in I$ if and only if $a \in I$ and $b \in I$. A dual ideal is a subset $D$ such that $a \cap b \in D$ if and only if $a \in D$ and $b \in D$. An ideal $I$ is called *completely irreducible* if it is not the intersection of ideals distinct from $I$; similarly a completely irreducible dual ideal is not the intersection of dual ideals distinct from it. The entire lattice and the empty set are irreducible ideals if the lattice has a smallest element. For partially ordered sets that are not lattices, ideals

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may be defined without reference to joins and meets (9). It has been shown that for a chain the completely irreducible dual ideals and ideals are just the open rays $(a, +\infty)$ and $(-\infty, b)$ consisting of elements $x > a$ and $x < b$ respectively.

In the *direct product* $P = \prod_j P_j$ of partially ordered sets $P_j$, the order relation is defined coordinatewise; $(a_j) \leqslant (b_j)$ if and only if $a_j \leqslant b_j$ for every index $j$. If the sets $P_j$ are lattices, then $P$ is a lattice and $(a_j) \cup (b_j) = (a_j \cup b_j)$; similarly for the meet operation, $(a_j) \cap (b_j) = (a_j \cap b_j)$.

The relationship between the ideals and dual ideals of a product of lattices $L = \prod_j L_j$ and those of the factors $L_j$ determines the relationship between their ideal topologies. This relationship is described in the following theorem:

THEOREM 1. *If a lattice $L$ is the product of lattices $\{L_j\}$, then every product of ideals of the factors $L_j$ is an ideal of $L$, and every product of dual ideals is a dual ideal of the product. In a finite product of lattices, every ideal is a product of ideals, and every dual ideal is a product of dual ideals.*

*Proof.* Suppose that for every index $j$ an ideal $I_j$ of the lattice $L_j$ is selected, and consider the product $I = \prod_j I_j$. Let $a = (a_j)$ and $b = (b_j)$ be any two elements of the product lattice $L$. Then for every index $j$ we conclude that $a_j \cup b_j \in I_j$ if and only if $a_j \in I_j$ and $b_j \in I_j$, since $I_j$ is an ideal. Since joins in the product $L$ are defined coordinatewise, it follows that $a \cup b \in I$ if and only if $a \in I$ and $b \in I$. Hence $I$ is an ideal. Similarly, a product of dual ideals is a dual ideal.

Conversely, if $I$ is any ideal of the product lattice $L$, then the projection $I_j$ of $I$ on the factor lattice $L_j$ is an ideal of $L_j$. The set $I_j$ consists of all elements $a_j$ that are $j$-coordinates of elements $a$ of $I$. If $a_j$ and $b_j$ are elements of $I_j$, then they are $j$-coordinates of elements $a$ and $b$ of $I$. Hence $a_j \cup b_j$ is the $j$-coordinate of the element $a \cup b$, which is in $I$ since $I$ is an ideal.

On the other hand if $a_j \cup b_j = c_j$ is any element of the projection $I_j$, then it is the $j$-coordinate of an element $c$ of $I$. If we call $a$ and $b$ the elements of $L$ obtained by replacing $c_j$ by $a_j$ and $b_j$ in $c$ respectively, then it is clear that $a \cup b = c$. Since $I$ is an ideal and $c \in I$, it follows that $a \in I$ and $b \in I$. Hence $a_j \in I_j$ and $b_j \in I_j$, which shows that the projection $I_j$ is an ideal of $L_j$. In the same way it can be shown that the projection $D_j$ of a dual ideal of the product lattice $L$ is a dual ideal of the lattice $L_j$.

Now any subset of a product is contained in the product of its projections. If $I$ is an ideal of the product $L = \prod_j L_j$ of a finite number of lattices $L_j$, we must show conversely that $I$ contains the product of its projections. It will follow that $I$ is equal to the product of its projections, and hence to a product of ideals.

Let $a = (a_j)$ be any element of the product $\prod_j I_j$ of the projections $I_j$ of the ideal $I$ on the lattice $L_j$. Let $b_j$ be the element of $I$ which has $a_j$ as its $j$-coordinate, and let $b = \bigcup_j b_j$. Then $b \in I$, since $b$ is the join of a finite number of elements of $I$. Clearly $a \leqslant b$; hence $a \in I$ also, since $I$ is an ideal and contains all elements contained in one of its members. This shows that $I$ is a product of ideals. In the same way it can be shown that every dual ideal of a finite product of lattices is a product of dual ideals. This completes the proof.

It is interesting to note that an ideal in the product of infinitely many lattices is not necessarily a product of ideals, but may be contained in the product of its projections without being equal to this product. For example, consider the Boolean algebra $B$ consisting of all subsets of a countably infinite set. Then $B$ can be represented as the product $\prod_j T_j$ of a countable infinity of copies $T_j$ of the two-element lattice $T$. Let $I$ be any non-principal maximal ideal of $B$. Since $I$ is non-principal, every projection $I_j$ of $I$ is identical with $T_j$. Hence $I \neq B = \prod_j T_j$. It follows that $I$ is not a product of ideals, since it is not the product of its projections. This is related to the fact that the ideal topology of the product of infinitely many lattices contains, but is not in general identical with, the cartesian product of the ideal topologies of the lattices.

THEOREM 2. *The ideal topology of the direct product of a finite number of lattices is equivalent to the cartesian product of the ideal topologies of the lattices.*

*Proof.* Let $L$ be the product of a finite number of lattices $L_j$. It follows from Theorem 1 that every ideal $I$ of $L$ is a product of ideals $I_j$ of $L_j$. If $I$ is a completely irreducible ideal, it is clear that each of its factor ideals $I_j$ is completely irreducible; for if $I_j$ were an intersection of ideals of $L_j$ all distinct from $I_j$, then $I$ could also be represented as an intersection of ideals of $L$ distinct from $I$.

Likewise, if $I$ is completely irreducible, then at most one of its factor ideals $I_j$ is a proper ideal, that is, distinct from $L_j$. For if two such factors $I_1$ and $I_2$ were proper ideals, then $I$ could be represented as an intersection of the two distinct products, obtained from the product representation of $I$ by replacing $I_1$ by $L_1$ and $I_2$ by $L_2$ respectively. It follows that the only completely irreducible ideals of $L$ are $L$ itself and products of completely irreducible ideals of $L_j$, all but one of which are equal to $L_j$.

Similarly, the completely irreducible dual ideals of $L$ consist of $L$ itself and of products of completely irreducible dual ideals of $L_j$, only one of which is a proper dual ideal.

Now we can examine the relationship between the ideal topology of $L$ and its cartesian topology. A base for the open sets in the ideal topology of the factor lattice $L_j$ consists of finite intersections of completely irreducible ideals and dual ideals of $L_j$. A base for the open sets in the cartesian product topology of $L$ consists of products of these basic open sets in each of the lattices $L_j$. A base for the open sets of $L$ in its ideal topology consists of finite intersections of completely irreducible ideals and dual ideals of $L$. We have just given the representation of these ideals and dual ideals of $L$ as products of ideals and dual ideals of the lattices $L_j$. It follows from this representation that the two collections of basic open sets of $L$ are the same. Hence the topologies are equivalent. This completes the proof.

It is easily verified that the ideal topology of a chain is equivalent to its intrinsic topology, since in both topologies the open rays $(a, +\infty)$ and $(-\infty, b)$ form a subbase for the open sets. It follows from Theorem 2 that the ideal topology of a lattice $L$ which is a finite product of chains is equivalent to the topology of $L$ as a cartesian product of the chains with their intrinsic topologies. In particular, it follows that the ideal topology of the euclidean space $R^k$, considered as a lattice which is a direct product of chains, is equivalent to the usual topology of $R^k$ as a metric space.

Theorem 2 does not extend to infinite products of lattices. The ideal topology of an infinite product contains the cartesian topology, since every basic open set in the cartesian topology is also a basic open set in the ideal topology of the product. This follows from Theorem 1.

However, in the Boolean algebra $B$ consisting of all subsets of a countably infinite set $N$, one can exhibit a subset of $B$ which is closed in the ideal topology of $B$ but not closed in $B$ considered as the cartesian product of countably many two-element lattices that are discrete spaces in the ideal topology. In fact, let $F$ be the ideal consisting of all finite subsets of $N$. Extend $F$ by Zorn's lemma to be a maximal proper ideal $I$ of $B$. Then $I$ is a completely irreducible ideal, and its complement is a completely irreducible dual ideal of $B$. It follows that $I$ is a closed set in the ideal topology of $B$. (It is also an open set.)

It is well known that $B$ in its cartesian product topology is homeomorphic to Cantor's nowhere dense perfect set, whose elements are represented by infinite ternary expansions containing only the digits 0 and 2. The finite subsets of $N$ have only a finite number of digits 2 in their expansions, and they correspond to the right-hand end points of the complementary middle-third intervals. The set $F$ is clearly dense in $B$ in the cartesian topology, which is the usual metric topology of the Cantor set. Hence the set $I$, which contains $F$, is dense in $B$. Since it is a proper subset of $B$, it is not a closed set in the cartesian topology. However, it is closed in the ideal topology. Hence the two topologies are distinct.

**3. The interval topology of products.** The interval topology of a partially ordered set or lattice is obtained by taking as a subbase for the closed sets the *closed rays* $[a, +\infty) = \{x: x \geqslant a\}$ and $(-\infty, b] = \{x: x \leqslant b\}$. The *closed intervals* $[a, b] = \{x: a \leqslant x \leqslant b\}$ are also closed in this topology, which accounts for the name. We shall show later that the closed rays and intervals are also closed in the ideal topology and the new interval topology.

The interval topology is simply defined and has some useful properties. For chains it is equivalent to the intrinsic topology. It is known that every chain is a completely normal Hausdorff space in its intrinsic topology **(7)**. It was proved in **(8)** that a complete lattice is compact in its interval topology. This topology has been used in the study of lattice-ordered groups and semigroups **(11, 17)**.

However, it was pointed out by Birkhoff, Rennie, and Northam that a product of chains without extreme elements, such as a euclidean space $R^k$, is not a Hausdorff space in its interval topology. It is shown in **(14)** that a Boolean algebra is not a Hausdorff space in its interval topology unless it is atomic. The ideal topology and the new interval topology were introduced in order to remedy some of the defects of the interval topology.

It was stated incorrectly in the paper **(8)** by Frink, in which the interval topology was first defined, that for a product of lattices the interval topology is equivalent to the cartesian product of the interval topologies of the factors. This is true for bounded lattices and in certain other cases, but not in general. We now derive some theorems about the interval topology of products.

THEOREM 3. *The interval topology of the direct product of a finite or infinite number of bounded lattices is equivalent to the cartesian product of the interval topologies of the lattices.*

*Proof.* Let $L$ be the product of the lattices $L_j$, each with smallest element $0_j$ and largest element $1_j$. Clearly $L$ has also a smallest element $0 = (0_j)$ and $1 = (1_j)$. In a lattice with 0 and 1 elements, every closed ray is also a closed interval, since $[a, +\infty) = [a, 1]$ and $(-\infty, b] = [0, b]$. Hence in such a lattice the closed intervals $[a, b]$ form a subbase for the closed sets in the interval topology. The entire lattice is also such a closed interval, namely $[0, 1]$.

It is clear that any closed interval $[a, b]$ of $L$ is a product of the closed intervals $[a_j, b_j]$, where $a = (a_j)$ and $b = (b_j)$. Conversely, any such product of closed intervals of $L_j$ is a closed interval of $L$. Hence the closed intervals of the product lattice $L$ form a subbase for the closed sets in the cartesian product topology of $L$.

This follows from the fact, which is easily verified, that a subbase for the closed sets in the cartesian product topology of a product of topological spaces $X_j$ is obtained by taking all products of subbasic closed sets of the spaces $X_j$, provided that all, or all except a finite number of the subbases for the spaces $X_j$, contain the space $X_j$ as an element. If infinitely many of the subbases for $X_j$ do not contain $X_j$ as a member, then it can be seen that the product space $X$ is not the union of a finite number of products of subbasic closed sets, as it should be.

Since the closed intervals of the product lattice $L$ also form a subbase for the closed sets of the interval topology of $L$, this topology is equivalent to the cartesian product topology. This completes the proof of Theorem 3.

It is easily verified that the interval topology of a direct product $L$ of lattices $L_j$ is always contained in the cartesian product of the lattices $L_j$. This follows from the fact that an upper closed ray $[a, +\infty)$ of the product is a product of upper closed rays $[a_j, +\infty)$ of the factors; similarly for lower closed rays. The inclusion here is reversed for the ideal topology, since the ideal topology of a product contains, instead of being contained in, the cartesian product of the ideal topologies.

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The interval topology of a product of only two lattices $L_1$ and $L_2$ may differ from the cartesian product of the two interval topologies, if $L_1$ has no largest element, and $L_2$ has no smallest element. This is connected with the fact that the product of an upper closed ray of $L_1$ and a lower closed ray of $L_2$ is neither an upper nor a lower closed ray of the product lattice. In fact, such a product of upper and lower closed rays may not be contained in a finite union of closed rays of the product. The following theorem illustrates this situation.

THEOREM 4. *The interval topology of a direct product $L$ of lattices $L_j$ is contained in, but distinct from, the cartesian product of the interval topologies of $L_j$, whenever at least one of the lattices $L_1$ has no largest element, and another lattice $L_2$ has no smallest element.*

*Proof.* Let $c_1$ be any element of $L_1$ and $c_2$ be any element of $L_2$ and let the set $A$ of $L$ consist of all $x$ of $L$ such that $x_1 \geqslant c_1$ and $x_2 \leqslant c_2$, where $x_1$ and $x_2$ are the $L_1$ and $L_2$ coordinates of $x$ respectively. Then the set $A$ is closed in the cartesian product topology of $L$, since it is the product of a closed ray in $L_1$, a closed ray in $L_2$, and closed sets consisting of the entire lattice $L_j$ for indices $j$ distinct from 1 and 2.

If the set $A$ were also closed in the interval topology of $L$, it would be contained in a finite union of upper and lower closed rays of $L$, and hence in the union of one upper closed ray $[a, +\infty)$ and one lower closed ray $(-\infty, b]$, since any finite union of upper closed rays of a lattice is contained in a single upper closed ray, and any finite union of lower closed rays is contained in a single lower closed ray.

Let $x$ be any element of $A$, and call the $L_1$ and $L_2$ coordinates of this element $x_1$ and $x_2$, and let the corresponding coordinates of $a$ and $b$ be $a_1$, $b_1$ and $a_2$, $b_2$ respectively. Since either $x \geqslant a$ or $x \leqslant b$, it follows that either $x_1 \leqslant b_1$ for all $x$ in $A$, or $a_2 \leqslant x_2$ for all $x$ in $A$. This would require that either $L_1$ has a largest element, namely $b_1$, or $L_2$ has a smallest element, namely $a_2$, contrary to assumption. This is because an upper bound of an upper closed ray of a lattice is necessarily an upper bound of the lattice, and a lower bound of a lower closed ray is a lower bound of the lattice. This contradiction shows that the set $A$ is not closed in the interval topology of $L$, and the two topologies are different. This completes the proof.

In order that the interval topology of a product of lattices be equivalent to the cartesian product of the interval topologies, it is not necessary that all the lattices have both largest and smallest elements, as in Theorem 3. We now give some other cases where the two topologies are equivalent.

THEOREM 5. *The interval topology of the product $P$ of any chain $C$, with or without extreme elements, and any bounded lattice $L$, is equivalent to the cartesian product of the interval topologies of $C$ and $L$.*

*Proof.* Let $A$ be any subbasic closed set in the cartesian product topology of $P$, of the form $R \times J$, where $R = [a, +\infty)$ is an upper closed ray of the chain

$C$, and $J = [c, d]$ is a closed interval of the bounded lattice $L$. We wish to show that $A$ is also a closed set in the interval topology of $P$. For every $x \geqslant a$ in $C$ we define the set

$$A_x = [a, x] \times [c, d] \cup [x, +\infty) \times [c, +\infty).$$

$A_x$ is the union of a closed interval of $P$ and an upper closed ray of $P$; hence it is closed in the interval topology of $P$.

It is clear that $A \subset A_x$ for every $x \geqslant a$ in $C$. Since $C$ is a chain, it can be seen that $A = \bigcap_x A_x$. Hence $A$, as an intersection of closed sets in the interval topology, is closed in this topology. Similarly a subbasic set of the cartesian topology of $P$ which is a product of a lower closed ray of $C$ and a closed interval of $L$ is closed in the interval topology of $P$. Hence the two topologies are equivalent. This completes the proof.

THEOREM 6. *The interval topology of the product of a finite number of chains, each with a smallest element, or each with a largest element, is equivalent to the cartesian product of the interval topologies of the chains.*

*Proof.* We give the proof in detail only for the product $P$ of two chains $C_1$ and $C_2$ with smallest elements $0_1$ and $0_2$, since the proof in the general case is similar. It is sufficient to show that every subbasic closed set of $P$ in the cartesian topology is also closed in the interval topology of $P$.

Such a subbasic closed set is the product of closed rays in $C_1$ and $C_2$. If both rays are upper rays or lower rays, their product is a closed ray of $P$. It remains to consider the product $A = R_1 \times R_2$ of an upper ray of one chain, say $C_1$, and a lower ray of the other chain $C_2$. Let the upper ray be $[a_1, +\infty)$. The lower ray of $C_2$ is a closed interval $[0_2, b_2]$, since $C_2$ has a least element. If $x$ is any element of $C_1$ such that $x \geqslant a_1$, let $A_x$ be the union of the closed interval $[a_1, x] \times [0_2, b_2]$ of $P$, and the closed ray $[x, +\infty) \times [0_2, +\infty)$. The set $A_x$ is closed in the interval topology of $P$. Since $C_1$ is a chain, it follows that $A = \bigcap_x A_x$. Hence $A$, as an intersection of closed sets, is closed in the interval topology of $P$. This completes the proof of Theorem 6.

Theorem 6 shows that for certain lattices, the interval topology and the ideal topology are equivalent. For a finite product of chains with neither largest nor smallest element, they are distinct. In general, the ideal topology is larger than the interval topology; that is, it has more closed sets and open sets. This was proved by A. J. Ward **(16)**, who showed that every closed ray of a lattice $L$ is a closed set in the ideal topology of $L$.

**4. The new interval topology of products.** The new interval topology was introduced by Birkhoff **(5)**. To obtain it, one first defines the *closed bounded sets* $B$ of a partially ordered set to be the intersections of finite unions of closed intervals $[a, b]$. One then defines a set $F$ to be *closed* if its intersection $F \cap B$ with every closed bounded set $B$ is a closed bounded set. It is easily verified that a set $F$ is closed if its intersection $F \cap J$ with every interval $J = [a, b]$ is a closed bounded set.

This method of defining a topology is a special case of the procedure of J. W. Alexander **(1)**, who starts with any family $C$ of sets closed under intersection and finite union. A set is closed in Alexander's sense if its intersection with any member of $C$ is a member of $C$. In a $k$-space, for example, the set $C$ consists of the compact sets of the space, and a set is closed if its intersection with every compact set is compact.

Birkhoff has shown that every conditionally complete lattice is a $k$-space in its new interval topology **(5)**. It does not seem to be known whether the cartesian product of two $k$-spaces is always a $k$-space, although it has been shown that the cartesian product of an uncountable infinity of $k$-spaces is not necessarily a $k$-space.

Birkhoff studies the new interval topology for partially ordered sets $D$ which are *dually directed*. This means that every two elements of $D$ have a common upper bound and a common lower bound. Lattices, and in particular chains, are always dually directed. Some of the relationships between the interval topology and the new interval topology have been investigated in the papers **(2, 5)**. For the convenience of the reader, we exhibit these relationships in the form of a theorem, not all of which is new.

THEOREM 7. *Every set that is closed in the interval topology of a lattice $L$ is also closed in the new interval topology of $L$. For chains and for bounded lattices the two topologies are equivalent. Whenever the two topologies are equivalent, $L$ is the union of two closed rays.*

*Proof*. Every closed ray of $L$ is closed in the new interval topology, since the intersection of a closed ray $[a, +\infty)$ or $(-\infty, b]$ with a closed interval $[c, d]$ is always either empty, a singleton, or a closed interval, and hence always a closed bounded set. Since any set closed in the interval topology is an intersection of finite unions of closed rays, it is also closed in the new interval topology.

For bounded lattices the two topologies are equivalent. As Birkhoff showed, this follows from the fact that every closed ray in a bounded lattice is also a closed interval. That the two topologies are equivalent for chains is easily verified, and was shown in **(2)**.

The entire lattice $L$ is always a closed set in the new interval topology, since its intersection with a closed bounded set $B$ is $B$. If the two interval topologies are equivalent, then $L$ is also a closed set in the interval topology. Hence $L$ is the union of a finite number of upper and lower closed rays. In a lattice, the join of a finite number of upper closed rays is a single upper closed ray; likewise for lower closed rays. Hence $L$ is the union of two closed rays, one upper and one lower. This completes the proof.

It is not known whether the new interval topology of a finite product of lattices is always equivalent to the cartesian product of the new interval topologies. For some infinite direct products of lattices the topologies are distinct, as we shall show. The next theorem indicates that the new interval topology of the product of any number of lattices at least always includes the cartesian product topology.

THEOREM 8. *If $L$ is the direct product of any number of lattices $L_j$, then every set $F$ of $L$ which is closed in the cartesian product of the new interval topologies of the lattices $L_j$ is also a closed set in the new interval topology of $L$.*

*Proof.* A subbase for the closed sets in the cartesian product topology of $L$ consists of products of the form $F = F_k \times P$, where $P = \prod_{j \neq k} L_j$ and $F_k$ is any set closed in the new interval topology of $L_k$. The set $F$ will be closed in the new interval topology of $L$ if its intersection with every closed interval $J$ of $L$ is a closed bounded set of $L$. This intersection $F \cap J$ has the form $E = C_k \times P_k$, where $C_k = F_k \cap J_k$, $J = \prod_j J_j$, and $P_k = \prod_{j \neq k} J_j$. Since $F_k$ is closed in the new interval topology of $L_k$, then $C_k$ is a closed bounded set of $L_k$. Hence $F \cap J$ is the product of a closed bounded set and a product of closed intervals, and is thus a closed bounded set in the new interval topology of $L$.

It follows that $F$ is also closed in the new interval topology of $L$. Since all sets of a subbase for the closed sets in the cartesian product are closed in the new interval topology of $L$, so are all closed sets of the cartesian product. This completes the proof.

THEOREM 9. *The new interval topology of the product $C$ of a finite number of chains $C_j$ is equivalent to the cartesian product of the new interval topologies of $C_j$.*

*Proof.* By Theorem 8 it is sufficient to show that any set $F$ that is closed in the new interval topology of $C$ is also closed in the cartesian product topology. Let $F$ be any such set, and let $x = (x_j)$ be any element of $C$ not in $F$. We must show that $x$ has a neighbourhood in the cartesian topology which is disjoint from $F$.

If $J$ is any closed interval of $C$, then $F \cap J$ is a closed bounded set of $C$. If none of the coordinates $x_j$ of the element $x$ is an extreme element of the chain $C_j$, then there will exist a closed interval $J = [a, b]$ of $C$ containing $x$, and such that $a_j < x_j < b_j$ for every index $j$. Consider the intersection $F \cap J$. As a closed bounded set it is the intersection of finite unions of closed intervals of $C$. Hence it is closed in the cartesian product topology of $J$ considered as the direct product of a finite number of closed intervals of the chains $C_j$.

Since $x$ is not in $F \cap J$, it follows that there is a basic open set in the cartesian topology of $J$, containing $x$, and disjoint from $F \cap J$. Such a basic open set $G$ is the product of open intervals of the chains $C_j$ such as $(c_j, d_j)$, where $a_j \leqslant c_j < d_j \leqslant b_j$. If a coordinate $x_j$ of the element $x$ is a largest or smallest element of the chain $C_j$, then the definition of the open set $G$ must be modified by replacing the open interval $(c_j, d_j)$ by an appropriate open ray $(c_j, \infty)$ or $(-\infty, d_j)$.

Since $G$ is disjoint from $F \cap J$ and $G \cap J$, it follows that $G$ is disjoint from $F$. Since for every element $x$ not in $F$ there exists a set $G$ open in the cartesian topology of $C$, containing $x$, and disjoint from $F$, it follows that $F$ is also closed in the cartesian topology. This completes the proof.

Theorem 9 does not extend to products of infinitely many chains. This follows from an example given in (**12**, p. 240), of a lattice $L$ that is the direct product of uncountably many copies of the real line $R$. In its topology as a cartesian product of chains $R$ with the intrinsic topology, or the equivalent new interval topology, $L$ is not a $k$-space. In fact, as Kelley shows, there exists in $L$ a set $A$ that is not closed in the cartesian topology, but whose intersection $A \cap C$ with every compact set $C$ is compact.

But $L$ is a conditionally complete lattice, and therefore $L$ is a $k$-space in its new interval topology. As Birkhoff has shown (**5**), a subset of a conditionally complete lattice is closed and bounded if and only if it is compact in the new interval topology. It follows that every conditionally complete lattice is a $k$-space in its new interval topology. Since $L$ is not a $k$-space in its cartesian topology, the two topologies are distinct.

THEOREM 10. *In any finite product of chains the ideal topology is equivalent to the new interval topology, and each is equivalent to the cartesian product of the intrinsic topologies of the chains. This topology is that of a completely regular Hausdorff space.*

*Proof.* This follows from Theorems 2 and 9, and from the fact that for chains the ideal topology and the new interval topology are equivalent to the intrinsic topology; see (**2**). It is known that every chain is a completely normal Hausdorff space in its intrinsic topology. Hence the cartesian topology of a finite product of chains is that of a completely regular Hausdorff space. This completes the proof. Note that it follows from Theorem 10 that both the ideal topology and the new interval topology of the euclidean space $R^k$ are equivalent to its usual topology as a metric space. It is not known whether every product of chains is a normal space in its cartesian topology, although it is known that not every such product is completely normal.

**5. Conclusion.** The ideal topology and the new interval topology both include the interval topology. In some lattices the ideal topology includes the new interval topology. We have not proved that this is always the case. It is not known whether the new interval topology of a finite product of lattices is always equivalent to the cartesian product of the new interval topologies. We have proved this only for chains.

It is known that a Boolean algebra is always a Hausdoıff space in its ideal topology, but unknown whether this is the case also for all distributive lattices. It might be interesting to modify the new interval topology by defining a set in a lattice to be closed if its intersection with every closed ray is closed in the interval topology.

Many of our results can be generalized from lattices to partially ordered sets. In the case of the ideal topology, this would require extending the notion of ideal from lattices to more general partially ordered sets, as is done by Frink (9). Recently J. Mayer and M. Novotny, in a paper entitled "On some topologies of products of ordered sets," which has not yet been published, have generalized the notion of ideal in partially ordered sets, and have obtained some results on products of ideal topologies analogous to our Theorem 2.

## References


**1.** J. W. Alexander, *On the concept of a topological space*, Proc. Nat. Acad. Sci., *25* (1939), 52–54.

**2.** R. A. Alo and Orrin Frink, *Topologies of chains* (unpublished).

**3.** R. M. Baer, *A characterization theorem for lattices with Hausdorff interval topology*, J. Math. Soc. Japan, *7* (1955), 177–181.

**4.** Garrett Birkhoff, *Lattice theory*, rev. ed. (New York, 1948).

**5.** ——— *A new interval topology for dually directed sets*, Revista Mat. y Fis. Teorica Tucuman, *14* (1962), 325–331.

**6.** Garrett Birkhoff and Orrin Frink, *Representations of lattices by sets*, Trans. Amer. Math. Soc., *64* (1948), 299–316.

**7.** N. Bourbaki, *Eléments de mathématique*, III, *Topologie générale*, chap. IX (Paris, 1948).

**8.** Orrin Frink, *Topology in lattices*, Trans. Amer. Math. Soc., *51* (1942), 568–582.

**9.** ——— *Ideals in partially ordered sets*, Amer. Math. Monthly, *61* (1954), 223–233.

**10.** David Gale, *Compact sets of functions and function rings*, Proc. Amer. Math. Soc., *1* (1950), 303–308.

**11.** Charles Holland, *The interval topology of a certain l-group*, Czechoslovak Math. J., *15* (1965), 311–314.

**12.** J. L. Kelley, *General topology*, (New York, 1955).

**13.** T. Naito, *Lattices with P-ideal topologies*, Tôhoku Math. J., *12* (1960), 235–262.

**14.** E. S. Northam, *The interval topology of a lattice*, Proc. Amer. Math. Soc., *4* (1953), 825–827.

**15.** B. C. Rennie, *Lattices*, Proc. London Math. Soc., *52* (1951), 386–400.

**16.** A. J. Ward, *On relations between certain intrinsic topologies on a partially ordered set*, Proc. Cambridge Phil. Soc., *51* (1955), 254–261.

**17.** E. S. Wolk, *On the interval topology of an l-group*, Proc. Amer. Math. Soc., *12* (1961), 304–307.



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