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The $ER(2)$-cohomology of $B\mathbb{Z}/(2^q)$ and $\mathbb{CP}^n$

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Abstract. The $ER(2)$-cohomology of $B\mathbb{Z}/(2^q)$ and $\mathbb{CP}^n$ are computed along with the Atiyah–Hirzebruch spectral sequence for $ER(2)^*(\mathbb{CP}^\infty)$. This, along with other papers in this series, gives us the $ER(2)$-cohomology of all Eilenberg–MacLane spaces.

1 Introduction

We are concerned with only two cohomology theories in this paper. All our work is at the prime 2. First, we have the Johnson–Wilson theory, $E(2)^*(-)$, introduced in [JW73, Remark 5.13], with coefficients $E(2)^* = \mathbb{Z}_{(2)}[v_1, v_2^{\pm 1}]$ where the degree of v_1 is -2 and the degree of v_2 is -6.

Second, the Real Johnson–Wilson theory, $ER(2)^*(-)$, from [KW07a, Theorem 1.7] and [HK01, Theorem 4.1], is the main theory of interest. The theory $E(2)$ is complex orientable and it inherits a $\mathbb{Z}/(2)$-action from complex conjugation on MU, the spectrum for complex cobordism. The theory $ER(2)$ is the homotopy fixed points of the spectrum $E(2)$ under this action and is just the $n = 2$ analog of $ER(1) = KO_{(2)}$.

This paper is part of a series developing the generalized cohomology theory, $ER(2)^*(-)$ (and often $ER(n)^*(-)$), as a working tool for algebraic topologists. Interest in this comes from two directions. First, there is a close connection between $ER(2)$ and $TMF_0(3)$ (see [HM16, Corollary 4.17]), and second, $ER(2)$ has already proven useful in applications, particularly to non-immersions of real projective spaces, for example in [KW08a, Theorem 1.9], [KW08b, Theorem 1.4], and [Ban13, Theorem 4.1].

A great deal is known about $ER(2)$ already. In particular, we know the homology of the Omega spectrum for $ER(2)$ [KW07b, Theorem 1.2 and Section 2], and the homotopy type of the spaces in the Omega spectrum, [KW13, Theorems 1–4 and 1–6 and related discussion]. For most n, $ER(2)^*(\mathbb{RP}^n)$ has been computed, [KW08a, Theorems 13.2 and 13.3 for n even], [KW08b, Theorem 8.2 for $n = 16k+1$], and [Ban13, Theorem 3.1 for $n = 16k + 9$]. We also know $ER(n)^*(BO(q))$, [KW15, Theorem 1.1].

It is hard to put the results of this paper into proper context, because the context is constantly expanding. This paper is part of a much larger project developing the

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computability and applicability of $ER(n)^*(-)$. Computing

$$ER(n)^*\big(K(\mathbb{Z}/(2^q),j)\big) \quad \text{and} \quad ER(n)^*(K(\mathbb{Z},j+1))$$

are long term goals. In [KLW16, Theorem 1.3], the $ER(n)$-cohomology is computed for

$$K(\mathbb{Z},2k+1), \quad K(\mathbb{Z}/(2^q),2k), \quad \text{and} \quad K(\mathbb{Z}/(2),2k+1).$$

This paper can be seen as the first attack on the odd $K(\mathbb{Z}/(2^q),2k+1)$ cases, which are considerably more complicated than the even ones.

In the case of $ER(2)$, we know that

$$ER(2)^*\big(K(\mathbb{Z}/(2^q),j)\big) = 0 = ER(2)^*(\mathbb{Z},j+1) \quad \text{for } j > 2.$$

In [KLW16, Theorem 1.3], we compute

$$ER(2)^*(K(\mathbb{Z}/(2^q),2)) \quad \text{and} \quad ER(2)^*(\mathbb{Z},3).$$

This paper will be devoted to computing

$$ER(2)^*(B\mathbb{Z}/(2^q) = K(\mathbb{Z}/(2^q),1)) \quad \text{and} \quad ER(2)^*(\mathbb{CP}^n).$$

Together with the second author's computation of

$$ER(2)^*(\mathbb{CP}^\infty) = ER(2)^*(K(\mathbb{Z},2)),$$

this completely solves the problem of computing $ER(2)^*(-)$ for Eilenberg–MacLane spaces.

Numerous other examples are given in [KLW16]. Somewhere in this mix is a computation of

$$ER(2)^*(BU(q)) \quad \text{and} \quad ER(2)^*(\wedge^n\mathbb{CP}^\infty).$$

The main tool we use is the stable cofibration from [KW07a, Display 1.1]:

$$\Sigma^{17}ER(2) \xrightarrow{x} ER(2) \longrightarrow E(2)\ , \tag{1.1}$$

where $x \in ER(2)^{-17}$, $2x = 0$, and the second map is the homotopy fixed point inclusion. This gives rise to a Bockstein spectral sequence (BSS) with $E_1 = E(2)^*(X)$ and collapses after E_8, because the self map above has the property that $x^7 = 0$. Here x generalizes the class $\eta \in \pi_1(BO) = KO^{-1}$ where $\eta^3 = 0 = 2\eta$.

Because we use a spectral sequence to compute most of our results, the answers are often stated in terms of associated graded objects. In addition to computing the BSS for the spaces of interest, we describe the Atiyah–Hirzebruch spectral sequence (AHSS) for $\mathbb{CP}^\infty$ and $\mathbb{CP}^n$. Because there are maps of all of our spaces to $\mathbb{CP}^\infty$, the complete description of $ER(2)^*(\mathbb{CP}^\infty)$ in [Lor16] maps to our results and solves many of the extensions we leave alone.

The complete description of our results, with all the various x^i-torsion, is somewhat lengthy and will be presented in Section 3. However, there are some results that can be presented in a clean fashion and could be of the most interest. We state them here.

The coefficient ring $ER(2)^*$ has two special elements, $\hat{v}_1$ and $\hat{v}_2$, that map to $v_1v_2^{-3}$ of degree 16 and v_2^{-8} of degree 48, respectively in $E(2)^*$. The element $\hat{v}_2$ is the periodicity element in $ER(2)^*$, and so $ER(2)$ is periodic of period 48. The ring $ER(2)^*$ has

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a lot of interesting structure (see Appendix A, and [KW07b, Proposition 2.1]), but if we only look at elements in degrees multiples of 16, it simplifies dramatically to

$$ER(2)^{16*} = \mathbb{Z}_{(2)}[\hat{v}_1, \hat{v}_2^{\pm 1}].$$

Since $E(2)$ is complex orientable, we know that $E(2)^*(\mathbb{CP}^\infty) = E(2)^*[[u]]$ with the degree of u equal to 2. If we define $\hat{u} = v_2^3 u$ of degree -16, then

$$\hat{p}_1 \longrightarrow \hat{u}c(\hat{u}),$$

where $\hat{u}c(\hat{u}) \in E(2)^{-32}(\mathbb{CP}^\infty)$ and $\hat{p}_1 \in ER(2)^{-32}(\mathbb{CP}^\infty)$, where c comes from complex conjugation and $\hat{p}_1$ is a modified first Pontryagin class. Modulo filtrations, we use $\hat{p}_1 = -\hat{u}^2$.

The previously mentioned non-immersion results for real projective spaces obtained from $ER(2)$ came about by looking only at $ER(2)^{8*}(\mathbb{RP}^n)$. There, higher powers of a generating class existed than in $E(2)^*(-)$ (see [KW08a, Theorems 1.6] and ([KW08b, Theorems 1.1] for n even, [KW08b, Theorem 1.3] for $n = 16k + 1$, and [Ban13, Theorem 3.2] for $n = 16k + 9$). Something very similar happens here and can be extracted as a reasonably presentable theorem. For any complex orientable cohomology theory we have a first Pontryagin class, and its $k + 1$-st power will be zero in $\mathbb{CP}^{2k}$ and $\mathbb{CP}^{2k+1}$. Because $ER(2)$ is not complex orientable, we do not have this restriction and are often able to see higher powers of the first Pontryagin class, which also exists for this theory.

Theorem 1.2 *With $ER(2)^{16*} = \mathbb{Z}_{(2)}[\hat{v}_1, \hat{v}_2^{\pm 1}]$, then $ER(2)^{16*}(\mathbb{CP}^{8k+i}) =$*

$$\begin{array}{ll}
ER(2)^{16*}[\hat{p}_1]/(\hat{p}_1^{4k+1}) & i = 0 \\
ER(2)^{16*}[\hat{p}_1]/(\hat{p}_1^{4k+2}, 2\hat{p}_1^{4k+1}) & i = 1 \\
ER(2)^{16*}[\hat{p}_1]/(\hat{p}_1^{4k+3}, 2\hat{p}_1^{4k+2}, \hat{v}_1\hat{p}_1^{4k+2}) & i = 2 \\
ER(2)^{16*}[\hat{p}_1]/(\hat{p}_1^{4k+4}, 2\hat{p}_1^{4k+2}, 2\hat{p}_1^{4k+3}, \hat{v}_1\hat{p}_1^{4k+2}, \hat{v}_1\hat{p}_1^{4k+3}) & i = 3 \\
ER(2)^{16*}[\hat{p}_1]/(\hat{p}_1^{4k+4}, 2\hat{p}_1^{4k+3}, \hat{v}_1\hat{p}_1^{4k+3}) & i = 4 \\
ER(2)^{16*}[\hat{p}_1]/(\hat{p}_1^{4k+4}, 2\hat{p}_1^{4k+3}) & i = 5 \\
ER(2)^{16*}[\hat{p}_1]/(\hat{p}_1^{4k+4}) & i = 6 \\
ER(2)^{16*}[\hat{p}_1]/(\hat{p}_1^{4k+4}) & i = 7
\end{array}$$

$$ER(2)^{16*}(\mathbb{CP}^\infty) = ER(2)^{16*}[[\hat{p}_1]]$$

The BSS gives the x^i-torsion generators precisely, but we use a spectral sequence to compute the BSS, so the x^i-torsion generators we see are really for the associated graded object for our auxiliary spectral sequence. The complete answer for these spaces gets quite complicated, but to give some insight here, we will describe how the elements of $ER(2)^{16*}(-)$ are related to the x^i-torsion. We know that we can have at maximum, x^7-torsion, but, in fact, our typical case has only x, x^3, and x^7-torsion. In two cases, out of eight, for $\mathbb{CP}^n$, we get x^5-torsion generators.

Remark 1.3 The exotic higher powers of the first Pontryagin class all go to zero in $E(2)^*(\mathbb{CP}^n)$. As such, they are divisible by x and thus torsion elements because of the long exact sequence coming from (1.1) that gives the exact couple. In particular, for the $i = 2$ and 4 cases, the elements $\hat{p}_1^{4k+2}$ and $\hat{p}_1^{4k+3}$ are x^4 times an x^7-torsion generator. For the $i = 3$ case, $\hat{p}_1^{4k+2}$ is x^4 times an x^7-torsion generator and $\hat{p}_1^{4k+3}$ is x^6 times an x^7-torsion generator. For $i = 1$ and 5, the torsion classes, $\hat{p}_1^{4k+1}$ and $\hat{p}_1^{4k+3}$, are x^2 times x^5-torsion generators. On these classes, $\hat{v}_1^j$ is non-zero. They are all x^2 times x^3-torsion generators.

Theorem 1.4 *In the associated graded object used to compute the reduced $ER^{16*}(\mathbb{CP}^\infty)$, we have the following x^i torsion generators:*

$$\begin{array}{ll} x^1 & (2,\hat{v}_1)ER(2)^{16*}[\hat{p}_1]\{\hat{p}_1\}, \\ x^3 & \mathbb{Z}/(2)[\hat{v}_2^{\pm 1},\hat{p}_1]\{\hat{p}_1^2\}, \\ x^7 & \mathbb{Z}/(2)[\hat{v}_2^{\pm 1}]\{\hat{p}_1\}. \end{array}$$

We recall $\hat{u} = uv_2^3$ and use $\hat{F}$, the modified formal group law defined in the next section. The well-known result for $BP^*(-)$, [Lan70], implies

$$E(2)^*\big(B\mathbb{Z}/(2^q)\big) = E(2)^*[[\hat{u}]]/[2^q]_{\hat{F}}(\hat{u}).$$

However, in the case of $q = 1$, we also get, [KW08a, Theorem 3.2]:

$$ER(2)^*\big(B\mathbb{Z}/(2)\big) = ER(2)^*[[\hat{u}]]/[2]_{\hat{F}}(\hat{u}).$$

The map $B\mathbb{Z}/(2^q) \to B\mathbb{Z}/(2)$ takes $\hat{u}$ to the $[2^{q-1}](\hat{u})$ sequence in $E(2)$-cohomology and $\hat{u}$ to z (definition of z) in $ER(2)$-cohomology.

For a ring S, the notation $S\{a,b\}$ stands for the free S-module on generators a and b.

Theorem 1.5

(i) *There is a filtration on $ER(2)^{16*}(B\mathbb{Z}/(2^q))$ such that the associated graded object is*

$$ER(2)^{16*}[[\hat{p}_1]]/(2^q\hat{p}_1) \oplus ER(2)^{16*}[[\hat{p}_1]]/(2)\{2^{q-1}\hat{u}\},$$

where z is represented by $2^{q-1}\hat{u}$.

(ii) *The elements z and $\hat{p}_1$ generate $ER(2)^{16*}(B\mathbb{Z}/(2^q))$, which can be written in terms of $z\hat{p}_1^i$ and $\hat{p}_1^i$.*

(iii) *The map $ER(2)^{8*}(B\mathbb{Z}/(2^q)) \to E(2)^{8*}(B\mathbb{Z}/(2^q))$ is an injection.*

(iv) *The extension problems for $2z$, z^2, and $2^q\hat{p}_1$, can be solved in $E(2)^{16*}(B\mathbb{Z}/(2^q))$ using the series for $[2^q](\hat{u})$, $[2^{q-1}](\hat{u})$, and $c(\hat{u})$.*

The setup for the following all comes from the work of the second author in [Lor16]. We have a norm map:

$$E(2)^*\big(B\mathbb{Z}/(2^q)\big) \xrightarrow{N_*} ER(2)^*\big(B\mathbb{Z}/(2^q)\big).$$

Definition 1.6 We take a restricted norm, and let $\mathrm{im}(N_*^{\mathrm{res}})$ be the image of the composition:

$$\mathbb{Z}_{(2)}[\hat{v}_1, v_2^{\pm 2}][[\hat{u}c(\hat{u})]]\{\hat{u}, v_2\hat{u}, z\hat{u}, v_2 z\hat{u}\} \longrightarrow E(2)^*(B\mathbb{Z}/(2^q)) \xrightarrow{N_*} ER(2)^*(B\mathbb{Z}/(2^q))$$

where $z = [2^{q-1}](\hat{u})$.

Under the reduction $ER(2)^*(B\mathbb{Z}/(2^q)) \to E(2)^*(B\mathbb{Z}/(2^q))$,

$$N_*(y) \longrightarrow y + c(y).$$

Continuing from [Lor16, Lemma 10.1], the image of $\hat{u}$ is a power series in $\hat{p}_1$, $N_*(\hat{u}) = \xi(\hat{p}_1)$. In the statement of the next theorem we use the elements $\alpha_i \in ER(2)^{-12i}$ for $0 \le i < 4$, which are introduced in the next section. To simplify notation, let $\alpha_{\{0,1,2,3\}}z$ or $\alpha_{\{0-3\}}z$ denote $\{\alpha_0 z = 2z, \alpha_1 z, \alpha_2 z, \alpha_3 z\}$ and so forth. Similar to the second author's result for $ER(n)^*(\mathbb{CP}^\infty)$, [Lor16, Theorem 1.1], we have the following theorem.

Theorem 1.7 *There is a short exact sequence of modules over $ER(2)^*$*

$$0 \longrightarrow \operatorname{im}(N_*^{\text{res}}) \longrightarrow ER(2)^*(B\mathbb{Z}/(2^q)) \longrightarrow \frac{ER(2)^*[[\hat{p}_1, z]]}{(J)} \longrightarrow 0,$$

where (J) is the ideal generated by power series representing $\xi(\hat{p}_1)$, z^2, $\alpha_{\{0-3\}}z$, and $2^{q-1}\alpha_{\{0-3\}}\hat{p}_1$, all computable by algorithms described in the proof of Theorem 1.5.

Note that the last map is not a ring map.

The paper is organized as follows. We do some necessary preliminaries in Section 2, state our BSS results in Section 3. Next, in Section 4, we do our BSS computations for $\mathbb{CP}^\infty$, and in Section 5, our BSS computations for $B\mathbb{Z}/(2^q)$. After that we describe the AHSS for $\mathbb{CP}^\infty$ in Section 6, followed by our computation of the BSS for $\mathbb{CP}^n$ and the proof of Theorem 1.2 in Section 7. We finish with Theorem 1.7 in Section 8. We include an Appendix A, giving a table for $ER(2)^*$ in its $\mathbb{Z}/(48)$-graded form.

2 Preliminaries

There are many ways to describe $ER(2)^*$, but we will stick mainly with the description given in [KW15, Remark 3.4]. See also Appendix A.

Although not always convenient, we traditionally call $\hat{v}_1 \in ER(2)^*$, α. It has degree 16 and maps to $v_1 v_2^{-3} \in E(2)^*$. We also have elements α_i, $0 < i < 4$, with degree $-12i$. We often extend this notation to $\alpha_0 = 2$. These elements map to $2v_2^{2i} \in E(2)^*$. For the last non-torsion algebra generator, we have w of degree -8, which maps to $\hat{v}_1 v_2^4 = v_1 v_2 \in E(2)^*$.

Torsion is generated by the element $x \in ER(2)^{-17}$. It has $2x = 0$ and $x^7 = 0$. Keep in mind that $ER(2)^*$ is 48 periodic with periodicity element $\hat{v}_2$ (mapping to v_2^{-8}). We use, for efficient notation, $x^{3-6} = \{x^3, x^4, x^5, x^6\}$.

Fact 2.1 $ER(2)^*$ is

$$\mathbb{Z}_{(2)}[\hat{v}_1, \hat{v}_2^{\pm 1}]\{1, w, \alpha_1, \alpha_2, \alpha_3\} \quad \text{with} \quad 2w = \alpha\alpha_2 = \hat{v}_1\alpha_2$$

$$\mathbb{Z}/(2)[\hat{v}_1, \hat{v}_2^{\pm 1}]\{x^{1-2}, x^{1-2}w\} \qquad \mathbb{Z}/(2)[\hat{v}_2^{\pm 1}]\{x^{3-6}\}.$$

What makes $ER(2)^*(-)$ computable is the result from [KW07a] that tells us that the fibre of the fixed point inclusion, $ER(2) \longrightarrow E(2)$, is just $\Sigma^{17}ER(2)$ and that the map of $\Sigma^{17}ER(2)$ to $ER(2)$ is just x with $x^7 = 0$, *i.e.*, we have the stable cofibration sequence

(2.2)
$$\begin{array}{ccc} \Sigma^{17}ER(2) & \xrightarrow{\quad x \quad} & ER(2) \\ & {}_{\partial}\nwarrow \qquad \swarrow & \\ & E(2). & \end{array}$$

From this, we have an exact couple and a convergent BSS that begins with $E(2)^*(X)$ and where there can only be differentials d_1 through d_7.

We have used two versions of this spectral sequence in the past. In [KW15, Theorem 2.1] we used the truncated version that converges to $ER(2)^*(X)$, but in [KW08a, Theorem 4.2] and [KW08b] we used the untruncated version that converges to zero. Both versions contain the same information, but the designated writer for this paper prefers the one converging to zero because it gets cleaner as each differential is computed. The drawback, of course, is that one must go back to the differentials to reconstruct $ER(2)^*(X)$.

We give a brief summary of the BSS for computing $ER(2)^*(X)$ from $E(2)^*(X)$.

Theorem 2.3 ([KW08a][Theorem 4.2])

(i) *The exact couple* (2.2) *gives a spectral sequence, E_r, of $ER(2)^*$ modules, starting with*
$$E_1 \simeq E(2)^*(X) \quad \textit{and ending with} \quad E_8 = 0.$$
(ii) $d_1(y) = v_2^{-3}(1-c)(y)$, *where $c(v_i) = -v_i$, and c comes from complex conjugation.*
(iii) *The degree of d_r is $17r + 1$.*
(iv) *The targets of the d_r represent the x^r-torsion generators of $ER(2)^*(X)$.*

Definition 2.4 Let K_i be the kernel of x^i on $ER(2)^*(X)$ and let M_i be the image of K_i in $ER(2)^*(X)/(xER(2)^*(X)) \subset E(2)^*(X)$. We call $M_r/M_{r-1} \simeq$ image d_r the *x^r-torsion generators.*

Remark 2.5 All of our BSS's in this paper have only even degree elements, so we always have $d_2 = d_4 = d_6 = 0$.

For our purposes, it is important to know how this works for the cohomology of a point. The differential d_1 commutes with $\hat{v}_1$ and v_2^2. All that matters here is that $d_1(v_2) = 2v_2^{-2}$.

The E_2 term becomes $\mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 2}]$. We have that d_3 commutes with $\hat{v}_1$ and v_2^4, and $d_3(v_2^2) = \hat{v}_1 v_2^{-4}$.

This leaves us with only $\mathbb{Z}/(2)[v_2^{\pm 4}]$. We have that d_7 commutes with $v_2^8 = \hat{v}_2^{-1}$ and $d_7(v_2^4) = \hat{v}_2 v_2^{-8} = \hat{v}_2^2 = v_2^{-16}$, so $E_8 = 0$.

In terms of describing our $ER(2)^*$ using this approach, we see that the x-torsion is generated by $\mathbb{Z}_{(2)}[\hat{v}_1, v_2^{\pm 2}]\{2\}$, the x^3-torsion by $\mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 4}]\{\hat{v}_1\}$, and the x^7-torsion by $\mathbb{Z}/(2)[\hat{v}_2^{\pm 1}]$. The previous description of $ER(2)^*$ is easy to relate to this now. The x-torsion is given by $\mathbb{Z}_{(2)}[\hat{v}_1, \hat{v}_2^{\pm 1}]$ on the α_i, $0 \le i < 4$. The x^3-torsion is

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generated over $\mathbb{Z}/(2)[\hat{v}_1, \hat{v}_2^{\pm 1}]$ on $\hat{v}_1 = \alpha$ and w. Finally, the x^7-torsion is given by $\mathbb{Z}/(2)[\hat{v}_2^{\pm 1}]$.

Combining the x, x^3, and x^7-torsion, we find that

$$ER(2)^{16*} = \mathbb{Z}_{(2)}[\hat{v}_1, \hat{v}_2^{\pm 1}].$$

The theory $E(2)^*(-)$ is a complex orientable theory, and so $E(2)^*(\mathbb{CP}^\infty) = E(2)^*[[u]]$, where u is of degree 2. For reasons that will become apparent later, we want to "hat" this like we did $E(2)^*$. This follows [KW15, pp. 235–236] and [KW08a, Section 5]. The only adjustment needed here is to define $\hat{u} = uv_2^3$, of degree −16. For the purposes of our BSS, we write $E(2)^*(\mathbb{CP}^\infty) = E(2)^*[[\hat{u}]]$. Since v_2 is a unit, this is not a problem.

Remark 2.6 The standard formal group law for $E(2)$ is

$$F(x, y) = \sum a_{i,j} x^i y^j \qquad a_{i,j} \in E(2)^{-2(i+j-1)}$$

with the degrees of x and y equal to two. The element $F(x, y)$ also has degree two. For $a \in E(2)^{2i}$, define $\hat{a} = av_2^{3i}$. Also, let $\hat{x} = v_2^3 x$, $\hat{y} = v_2^3 y$, and $\hat{u} = uv_2^3$. Special cases are $\hat{v}_1 = v_1 v_2^{-3}$ and $\hat{v}_2 = v_2 v_2^{-9} = v_2^{-8}$. Now, define

$$\begin{aligned}\hat{F}(\hat{x}, \hat{y}) &= \sum \hat{a}_{i,j} \hat{x}^i \hat{y}^j = \sum a_{i,j} v_2^{-3(i+j-1)} x^i v_2^{3i} y^j v_2^{3j} \\ &= v_2^3 \sum a_{i,j} x^i y^j = v_2^3 F(x, y).\end{aligned}$$

We will need the complex conjugate of $\hat{u}$, $c(\hat{u})$. It has the defining property that $\hat{F}(\hat{u}, c(\hat{u})) = 0$.

The formal group law begins with

$$\hat{F}(\hat{x}, \hat{y}) = \hat{x} + \hat{y} + \hat{v}_1 \hat{x}\hat{y},$$

so $c(\hat{u})$ begins $-\hat{u} + \hat{v}_1 \hat{u}^2$. When we work mod 2, the formal group law begins

$$\hat{F}(\hat{x}, \hat{y}) = \hat{x} + \hat{y} + \hat{v}_1 \hat{x}\hat{y} + \hat{v}_1^2(\hat{x}^2\hat{y} + \hat{x}\hat{y}^2) + \hat{v}_2 \hat{x}^2 \hat{y}^2$$

and the corresponding computation mod 2 begins

$$c(\hat{u}) = \hat{u} + \hat{v}_1 \hat{u}^2 + \hat{v}_1^2 \hat{u}^3 + \hat{v}_2 \hat{u}^4$$

While we are working with the formal group law, we need another fact. Recall that $[2](\hat{x}) = \hat{F}(\hat{x}, \hat{x})$ and $[2^q](\hat{x}) = [2]([2^{q-1}(\hat{x})])$. When we set $0 = [2^q](\hat{x})$, we need to know that

$$0 = 2^q \hat{x} + 2^{q-1} \hat{v}_1 \hat{x}^2 \mod (\hat{x}^3).$$

This follows from a simple induction on q.

We collect the basics we need in the following lemma.

Lemma 2.7

$$\begin{aligned}c(\hat{u}) &= -\hat{u} + \hat{v}_1 \hat{u}^2 && \mod (\hat{u}^3); \\ c(\hat{u}) &= \hat{u} + \hat{v}_1 \hat{u}^2 + \hat{v}_1^2 \hat{u}^3 + \hat{v}_2 \hat{u}^4 && \mod (2, \hat{u}^5).\end{aligned}$$

If we set $0 = [2^q](\hat{u})$, then

$$0 = [2^q](\hat{u}) = 2^q \hat{u} + 2^{q-1} \hat{v}_1 \hat{u}^2 \qquad \mod (\hat{u}^3).$$

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There is a modified Pontryagin class, $\hat{p}_1 \in ER(2)^{-32}(\mathbb{CP}^\infty)$, which maps to $\hat{u}c(\hat{u}) \in E(2)^{-32}(\mathbb{CP}^\infty)$.

Filtering by powers of $\hat{u}$, $\hat{p}_1 = \hat{u}c(\hat{u}) = -\hat{u}^2$ in the associated graded object.

Proof The only things left to prove are the statements about $\hat{p}_1$. We refer the reader to [Lor16, Section 5]. The sign in the last statement follows immediately from the previous statements. ∎

3 Statement of Results for the BSS

We state all the results for the BSS for $ER(2)^*(-)$ for $\mathbb{CP}^\infty$, $B\mathbb{Z}/(2^q)$, and $\mathbb{CP}^n$. In each case there is a filtration on $ER(2)^*(X)$ such that we can identify the representatives of the x^i-torsion generators of $ER(2)^*(X)$ in the associated graded object. This comes about because we use a spectral sequence to compute the differentials in the BSS. Keep in mind that the element $\hat{p}_1$ is represented by $-\hat{u}^2$ in the associated graded object. Also keep in mind that we are working with the entire cohomology here, not just the degrees $16*$ discussed in the introduction.

Theorem 3.1 *There is a filtration on the reduced $ER(2)^*(\mathbb{CP}^\infty)$ such that we can identify the representatives of the x^i-torsion generators for $ER(2)^*(\mathbb{CP}^\infty)$ in the associated graded object as follows:*

The x^1-torsion generators are

$$\mathbb{Z}_{(2)}[\hat{v}_1, v_2^{\pm 2}, \hat{u}^2]\{2v_2\hat{u}, 2\hat{u}^2\} \quad \text{and} \quad \mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 2}, \hat{u}^2]\{\hat{v}_1\hat{u}^2\}.$$

The x^3-torsion generators are $\mathbb{Z}/(2)[v_2^{\pm 4}, \hat{u}^2]\{\hat{u}^4\}$.

The x^7-torsion generators are $\mathbb{Z}/(2)[\hat{v}_2^{\pm 1}]\{\hat{u}^2\}$.

Remark 3.2 The reader will note that there is an obvious extension in the x^1-torsion, *i.e.,* 2 times the elements on the right are in the module on the left. Honest x^r-torsion generators reduce to elements that are the image of d_r. Since we have filtered our spectral sequence, none of our x^r-torsion generators are honest, because we only see these images in the first filtration they show up. In the case of the x^1-torsion, we need two differentials in our filtration in order to compute d_1. The way we describe the result corresponds to the images of those two differentials.

Theorem 3.3 *There is a filtration on the reduced $ER(2)^*(B\mathbb{Z}/(2^q))$ such that we can identify the representatives of the x^i-torsion generators for $ER(2)^*(B\mathbb{Z}/(2^q))$ in the associated graded object as follows:*

The x^1-torsion generators are

$$\mathbb{Z}/(2^{q-1})[\hat{v}_1, v_2^{\pm 2}, \hat{u}^2]\{2v_2\hat{u}, 2\hat{u}^2\} \quad \text{and} \quad \mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 2}, \hat{u}^2]\{\hat{v}_1\hat{u}^2, 2^{q-1}\hat{v}_1\hat{u}^3\}.$$

The x^3-torsion generators are

$$\mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 4}]\{2^{q-1}\hat{v}_1\hat{u}\} \quad \text{and} \quad \mathbb{Z}/(2)[v_2^{\pm 4}, \hat{u}^2]\{\hat{u}^4, 2^{q-1}\hat{u}^5\}.$$

The x^7-torsion generators are $\mathbb{Z}/(2)[\hat{v}_2^{\pm 1}]\{2^{q-1}\hat{u}, \hat{u}^2, 2^{q-1}\hat{u}^3\}$.

Note that the x^7-torsion generators are just z, $\hat{p}_1$, and $z\hat{p}_1$.

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The case for $ER(2)^*(\mathbb{CP}^n)$ is significantly more complicated to state. We break it up into a series of theorems.

Theorem 3.4 *There is a filtration on the reduced $ER(2)^*(\mathbb{CP}^{2j})$ such that we can identify the representatives of the x and x^3-torsion generators for $ER(2)^*(\mathbb{CP}^{2j})$ in the associated graded object as follows:*

The x^1-torsion generators are

$$\mathbb{Z}_{(2)}[\hat{v}_1, v_2^{\pm 2}, \hat{u}^2]/(\hat{u}^{2j})\{2v_2\hat{u}, 2\hat{u}^2\} \quad \text{and} \quad \mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 2}, \hat{u}^2]/(\hat{u}^{2j})\{\hat{v}_1\hat{u}^2\}.$$

The x^3-torsion generators are $\mathbb{Z}/(2)[v_2^{\pm 4}, \hat{u}^2]/(\hat{u}^{2j-2})\{\hat{u}^4\}$.

Theorem 3.5 *There is a filtration on the reduced $ER(2)^*(\mathbb{CP}^{2j+1})$ such that we can identify the representatives of the x and x^3-torsion generators for $ER(2)^*(\mathbb{CP}^{2j+1})$ in the associated graded object as follows:*

The x^1-torsion generators are

$$\mathbb{Z}_{(2)}[\hat{v}_1, v_2^{\pm 2}, \hat{u}^2]/(\hat{u}^{2j+1})\{2v_2\hat{u}\}, \quad \mathbb{Z}_{(2)}[\hat{v}_1, v_2^{\pm 2}, \hat{u}^2]/(\hat{u}^{2j})\{2\hat{u}^2\}, \quad \text{and}$$
$$\mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 2}, \hat{u}^2]/(\hat{u}^{2j})\{\hat{v}_1\hat{u}^2\}.$$

The x^3-torsion generators are

$$\mathbb{Z}/(2)[v_2^{\pm 4}, \hat{u}^2]/(\hat{u}^{2j-2})\{\hat{u}^4\} \quad \text{and} \quad \mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 4}]\{\hat{v}_1 v_2^{2j+1}\hat{u}^{2j+1}\}.$$

Theorem 3.6 *There is a filtration on $ER(2)^*(\mathbb{CP}^{8k+i})$ such that we can identify the representatives of the x^5-torsion and x^7-torsion generators for the reduced $ER(2)^*(\mathbb{CP}^{8k+i})$ in the associated graded object as follows:*

For all i there are x^7-torsion generators $\mathbb{Z}/(2)[\hat{v}_2^{\pm 1}]\{\hat{u}^2\}$.

The x^5-torsion generators are

$$\text{for } i = 1: \quad \mathbb{Z}/(2)[v_2^{\pm 4}]\{v_2^5\hat{u}^{8k+1}\},$$
$$\text{for } i = 5: \quad \mathbb{Z}/(2)[v_2^{\pm 4}]\{v_2\hat{u}^{8k+5}\}.$$

The rest of the x^7-torsion generators are

$$\text{for } i = 0: \quad \mathbb{Z}/(2)[\hat{v}_2^{\pm 1}]\{v_2^6\hat{u}^{8k}\},$$
$$\text{for } i = 2: \quad \mathbb{Z}/(2)[\hat{v}_2^{\pm 1}]\{v_2^2\hat{u}^{8k+2}\},$$
$$\text{for } i = 3: \quad \mathbb{Z}/(2)[\hat{v}_2^{\pm 1}]\{v_2^2\hat{u}^{8k+2}, v_2^7\hat{u}^{8k+3}\},$$
$$\text{for } i = 4: \quad \mathbb{Z}/(2)[\hat{v}_2^{\pm 1}]\{v_2^2\hat{u}^{8k+4}\},$$
$$\text{for } i = 6: \quad \mathbb{Z}/(2)[\hat{v}_2^{\pm 1}]\{v_2^6\hat{u}^{8k+6}\},$$
$$\text{for } i = 7: \quad \mathbb{Z}/(2)[\hat{v}_2^{\pm 1}]\{v_2^6\hat{u}^{8k+6}, v_2^3\hat{u}^{8k+7}\}.$$

4 $ER(2)^*(\mathbb{CP}^\infty)$

In this section we give a quick and dirty computation of $ER(2)^*(\mathbb{CP}^\infty)$. This is done with complete detail and real finesse in [Lor16] for all $ER(n)$, but what we do here is enough for our purposes.

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We filter again and use an auxiliary spectral sequence to compute the BSS for $\mathbb{CP}^\infty$. As a result, even our BSS computation results are given in terms of an associated graded object. We will abuse the notation and continue to call these terms E_r.

Theorem 4.1 *Filtering $E(2)^*(\mathbb{CP}^\infty) = E(2)^*[[\hat{u}]]$ by powers of $\hat{u}$ we give E_r as an associated graded object of the actual E_r of the BSS for the reduced $ER(2)^*(\mathbb{CP}^\infty)$:*

$$E_1 = E(2)^*[\hat{u}]\{\hat{u}\}, \qquad E_2 = E_3 = \mathbb{Z}/(2)[v_2^{\pm 2}, \hat{u}^2]\{\hat{u}^2\},$$
$$E_4 = E_5 = E_6 = E_7 = \mathbb{Z}/(2)[v_2^{\pm 4}]\{\hat{u}^2\}, \qquad E_8 = 0.$$

Proof We are first going to compute a spectral sequence for computing d_1. This spectral sequence will collapse after the first two differentials, which we will call $d_{1,1}$ and $d_{1,2}$. Our spectral sequence for computing d_1 comes from filtering by powers of $\hat{u}$.

From the computation of d_1 for $ER(2)^*$, we know that d_1 commutes with v_2^2 and $\hat{v}_1$. Filtering by the powers of $\hat{u}$, we have $c(\hat{u}) = -\hat{u}$ (from Lemma 2.7) in the associated graded object, so

$$\begin{aligned} d_{1,1}(\hat{u}) &= v_2^{-3}(1-c)\hat{u} = v_2^{-3}(\hat{u} - (-1)\hat{u}) = 2v_2^{-3}\hat{u}. \\ d_{1,1}(\hat{u}^2) &= v_2^{-3}(1-c)\hat{u}^2 = v_2^{-3}(\hat{u}^2 - (-1)^2\hat{u}^2) = 0. \\ d_{1,1}(v_2\hat{u}) &= v_2^{-3}(1-c)v_2\hat{u} = v_2^{-3}(v_2\hat{u} - (-1)^2 v_2\hat{u}) = 0. \\ d_{1,1}(v_2\hat{u}^2) &= v_2^{-3}(1-c)v_2\hat{u}^2 = v_2^{-3}(v_2\hat{u}^2 - (-1)^3 v_2\hat{u}^2) = 2v_2^{-2}\hat{u}^2. \end{aligned}$$

We can now read off the first term of the x-torsion in Theorem 3.1. After taking the homology with respect to $d_{1,1}$, all we have left are

$$\mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 2}, \hat{u}^2]\{v_2\hat{u}\} \quad \text{and} \quad \mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 2}, \hat{u}^2]\{\hat{u}^2\}.$$

There is one more computation to do to finish off d_1. The second differential, $d_{1,2}$, in the spectral sequence to compute d_1 requires the use of not just $\hat{u}^j$ but $\hat{u}^{j+1}$. We are now working mod 2 so we have $c(\hat{u}) = \hat{u} + \hat{v}_1\hat{u}^2$ (from Lemma 2.7). Recall again that d_1 commutes with $\hat{v}_1$ and v_2^2. We have our second differential:

$$\begin{aligned} d_{1,2}(v_2\hat{u}) &= v_2^{-3}(1+c)v_2\hat{u} = v_2^{-3}(v_2\hat{u} + c(v_2\hat{u})) \\ &= v_2^{-3}(v_2\hat{u} + v_2(\hat{u} + \hat{v}_1\hat{u}^2)) = v_2^{-2}(\hat{v}_1\hat{u}^2), \\ d_{1,2}(\hat{u}^2) &= v_2^{-3}(1+c)\hat{u}^2 = v_2^{-3}(\hat{u}^2 + c(\hat{u}^2)) \\ &= v_2^{-3}(\hat{u}^2 + (\hat{u} + \hat{v}_1\hat{u}^2)^2) = v_2^{-3}(2\hat{u}^2 + 2\hat{v}_1\hat{u}^3) = 0. \end{aligned}$$

This gives us our final term of x-torsion in Theorem 3.1. All that remains after taking the homology with respect to $d_{1,2}$ is our stated E_2 term.

With the degree of v_2 equal to -6 and the degree of $\hat{u}$ equal to -16, we see that this is all in degrees multiples of 4, but the differential d_1 has degree 18, which equals 2 mod 4, so there can be no more to the differential d_1, so this is an associated graded version of E_2.

The differential d_2 is odd degree, so it is zero, and we have computed E_3. Although we have only computed an associated graded version of E_3, we know that $2x = 0$, and all x-torsion was detected by d_1. Consequently, the real E_3 is a $\mathbb{Z}/(2)$-vector space. Any extensions only involve $\hat{v}_1$.

We have that $\hat{v}_1$ acts trivially on our associated graded version of E_3, but it is not actually zero on E_3. We need to solve this extension problem. Because we know that $\hat{p}_1 \in ER(2)^*(\mathbb{CP}^\infty)$ maps to $\hat{u}c(\hat{u}) \in E(2)^*(\mathbb{CP}^\infty)$, $\hat{v}_1\hat{p}_1 \neq 0$, and so must have a representative in our $E_2 = E_3$.

Working mod 2 and mod $\hat{u}^5$, recall from Lemma 2.7 that we have

$$c(\hat{u}) = \hat{u} + \hat{v}_1\hat{u}^2 + \hat{v}_1^2\hat{u}^3 + \hat{v}_2\hat{u}^4.$$

Now take

$$d_1(v_2^3\hat{u}) = v_2^{-3}\big(v_2^3\hat{u} + v_2^3(\hat{u} + \hat{v}_1\hat{u}^2 + \hat{v}_1^2\hat{u}^3 + \hat{v}_2\hat{u}^4)\big) = \hat{v}_1\hat{u}^2 + \hat{v}_1^2\hat{u}^3 + \hat{v}_2\hat{u}^4.$$

This has to be zero in E_2 (mod 2 and $\hat{u}^5$). Now, in the first term, substitute

$$\hat{u} = c(\hat{u}) + \hat{v}_1\hat{u}^2 + \hat{v}_1^2\hat{u}^3$$

for one of the $\hat{u}$ to get

$$\begin{aligned} d_1(v_2^3\hat{u}) &= \hat{v}_1\hat{u}^2 + \hat{v}_1^2\hat{u}^3 + \hat{v}_2\hat{u}^4 \\ &= \hat{v}_1\hat{u}(c(\hat{u}) + \hat{v}_1\hat{u}^2 + \hat{v}_1^2\hat{u}^3) + \hat{v}_1^2\hat{u}^3 + \hat{v}_2\hat{u}^4 \\ &= \hat{v}_1\hat{u}c(\hat{u}) + \hat{v}_2\hat{u}^4 + \hat{v}_1^3\hat{u}^4. \end{aligned}$$

The last term here does not exist in our E_2 so must be represented in a higher filtration. Our representative for $\hat{v}_1\hat{p}_1$ is thus $\hat{v}_2\hat{u}^4$. Since $\hat{p}_1$ is represented by $\hat{u}^2$, we have

$$0 = \hat{v}_1\hat{u}^2 + \hat{v}_2\hat{u}^4. \tag{4.2}$$

We are ready to compute d_3 on $\mathbb{Z}/(2)[v_2^{\pm 2}, \hat{u}^2]\{\hat{u}^2\}$. We know that $\hat{u}c(\hat{u}) = \hat{p}_1$ is a permanent cycle from Lemma 2.7, but this is, mod $\hat{u}^5$, just $\hat{u}^2 + \hat{v}_1\hat{u}^3 + \hat{v}_1^2\hat{u}^4$. The last two terms would have to be represented in higher filtrations, so they are zero mod $\hat{u}^5$. So, modulo $\hat{u}^5$, we have that $d_3(\hat{u}^2) = 0$.

Compute

$$d_3(v_2^2\hat{u}^2) = d_3(v_2^2)\hat{u}^2 = \hat{v}_1 v_2^{-4}\hat{u}^2 = v_2^{-4}\hat{v}_2\hat{u}^4 = v_2^{-12}\hat{u}^4.$$

This gives the x^3-torsion of Theorem 3.1. Since d_3 commutes with v_2^4 and $\hat{u}^2$, the homology gives E_4 as stated.

Elements of E_4 are spaced out by the degree of v_2^4, or, -24. The differentials d_4, d_5, and d_6 have degrees 69, 86, and 103, and these are all non-zero mod 24, so these differentials are all trivial.

We know that $\hat{u}c(\hat{u})$ is a permanent cycle, so our differential must be on the v_2^4 as in the coefficients. We get

$$d_7(v_2^4\hat{u}^2) = \hat{v}_2 v_2^{-8}\hat{u}^2 = v_2^{-16}\hat{u}^2 = \hat{v}_2^2\hat{u}^2.$$

This gives our x^7-torsion for Theorem 3.1 and $E_8 = 0$. ∎

This also completes our description of $ER(2)^{16*}(\mathbb{CP}^\infty)$ in Theorems 1.2 and 1.4.

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5 $ER(2)^*(B\mathbb{Z}/(2^q))$

As mentioned in the introduction,

$$E(2)^*(B\mathbb{Z}/(2^q)) = E(2)^*[[\hat{u}]]/[2^q]_{\hat{F}}(\hat{u}),$$
$$ER(2)^*(B\mathbb{Z}/(2)) = ER(2)^*[[\hat{u}]]/[2]_{\hat{F}}(\hat{u}).$$

As we saw above, $\hat{u} \in E(2)^{-16}(\mathbb{CP}^\infty)$ is not a permanent cycle in the BSS for $\mathbb{CP}^\infty$, but this result for $\mathbb{RP}^\infty = B\mathbb{Z}/(2)$ shows that $\hat{u} \in E(2)^*(B\mathbb{Z}/(2))$ and all its powers must be permanent cycles in the BSS for $B\mathbb{Z}/(2)$. The BSS for this is described in [KW08a, Theorem 8.1] but redone here.

Theorem 5.1 *Filtering $E(2)^*(B\mathbb{Z}/(2^q)) = E(2)^*[[\hat{u}]]/([2^q](\hat{u}))$ by powers of $\hat{u}$ we give E_r as an associated graded object of the actual E_r of the reduced BSS for $ER(2)^*(B\mathbb{Z}/(2^q))$:*

$$E_1 = \mathbb{Z}/(2^q)[\hat{v}_1, v_2^{\pm 1}, \hat{u}]\{\hat{u}\},$$
$$E_2 = E_3 = \mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 2}]\{2^{q-1}\hat{u}\} \qquad \mathbb{Z}/(2)[v_2^{\pm 2}, \hat{u}^2]\{\hat{u}^2, 2^{q-1}\hat{u}^3\},$$
$$E_4 = E_5 = E_6 = E_7 = \mathbb{Z}/(2)[v_2^{\pm 4}]\{2^{q-1}\hat{u}, \hat{u}^2, 2^{q-1}\hat{u}^3\},$$
$$E_8 = 0.$$

Proof Since $[2^q](\hat{u}) = 2^q\hat{u} \bmod \hat{u}^2$, if we start our BSS for $ER(2)^*(B\mathbb{Z}/(2^q))$ not with $E(2)^*(B\mathbb{Z}/(2^q))$, but by filtering this by powers of $\hat{u}$, we get an associated graded version of E_1 as above.

We have a surjective map $E(2)^*(\mathbb{CP}^\infty) \to E(2)^*(B\mathbb{Z}/(2^q))$. In both cases we start by filtering by powers of $\hat{u}$, so this gives a map of associated graded rings. We use this filtration to compute d_1, and our first differential is inherited from $\mathbb{CP}^\infty$ giving the first part of the x-torsion of Theorem 3.3. Taking the homology with respect to this $d_{1,1}$ differential gives a very different answer from that for $\mathbb{CP}^\infty$. We get

$$\mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 2}, \hat{u}^2]\{v_2\hat{u}, 2^{q-1}\hat{u}, \hat{u}^2, 2^{q-1}v_2\hat{u}^2\}$$

We now need to compute the second differential in our spectral sequence for d_1, *i.e.*, we need to take into consideration $\hat{u}^j$ and $\hat{u}^{j+1}$. We need the solution to the extension problem on our generators given by the 2^q-series modulo $\hat{u}^3$ (from Lemma 2.7):

$$2(2^{q-1}\hat{u}) = 2^{q-1}\hat{v}_1\hat{u}^2 \quad \text{and} \quad 2(2^{q-1}\hat{u}^2) = 2^{q-1}\hat{v}_1\hat{u}^3.$$

We compute our $d_{1,2}$ on the generators with the continued understanding that the differential commutes with $\hat{v}_1$, v_2^2, and $\hat{u}^2$:

$$d_{1,2}(v_2\hat{u}) = v_2^{-3}(1+c)v_2\hat{u} = v_2^{-3}(v_2\hat{u} + v_2(\hat{u} + \hat{v}_1\hat{u}^2)) = v_2^{-2}\hat{v}_1\hat{u}^2,$$
$$d_{1,2}(2^{q-1}\hat{u}) = v_2^{-3}(1+c)2^{q-1}\hat{u} = 2^{q-1}v_2^{-3}(\hat{u} + (\hat{u} + \hat{v}_1\hat{u}^2))$$
$$= v_2^{-3}(2^q\hat{u} + 2^{q-1}\hat{v}_1\hat{u}^2) = v_2^{-3}(2^{q-1}\hat{v}_1\hat{u}^2 + 2^{q-1}\hat{v}_1\hat{u}^2) = 0,$$
$$d_{1,2}(\hat{u}^2) = 0,$$
$$d_{1,2}(2^{q-1}v_2\hat{u}^2) = 2^{q-1}\hat{u}^2 d_1(v_2) = 2^q v_2^{-2}\hat{u}^2 = 2^{q-1}\hat{v}_1 v_2^{-2}\hat{u}^3.$$

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This gives us the last of the x-torsion elements in Theorem 3.3. The homology after this $d_{1,2}$ is the E_2 stated in the theorem. The degrees of all the elements we have left are divisible by 4, and our d_1 is of degree 2 mod 4, so we have finished with our d_1.

The map $B\mathbb{Z}/(2^q) \to B\mathbb{Z}/(2)$ induces the map

$$E(2)^*\big(B\mathbb{Z}/(2)\big) \longrightarrow E(2)^*(B\mathbb{Z}/(2^q))$$

taking $\hat{u}$ to $[2^{q-1}](\hat{u}) \in E(2)^*(B\mathbb{Z}/(2^q))$. Consequently, this must be a permanent cycle. In our associated graded E_3, this means that $2^{q-1}\hat{u}$ has no differential. We have $\hat{u}^{2k}$ has no differential, and so we get that $2^{q-1}\hat{u}^{2k+1}$ has no differential. Consequently, the differential d_3 in our filtration is given by

$$d_3(2^{q-1}v_2^2\hat{u}^{2k+1}) = 2^{q-1}\hat{v}_1 v_2^{-4}\hat{u}^{2k+1} \quad \text{and} \quad d_3(v_2^2\hat{u}^{2k}) = \hat{v}_1 v_2^{-4}\hat{u}^{2k}.$$

In the first case, if $k = 0$, then $\hat{v}_1$ is there. If $k > 0$ or we look at the second case, we use the relation inherited from $\mathbb{CP}^\infty$ (from (4.2)): $\hat{v}_1\hat{u}^2 = \hat{v}_2\hat{u}^4$. This gives

$$2^{q-1}\hat{v}_1 v_2^{-4}\hat{u}^{2k+3} = 2^{q-1}v_2^{-4}\hat{v}_2\hat{u}^{2k+5} \quad \text{and} \quad \hat{v}_1 v_2^{-4}\hat{u}^{2k} = v_2^{-4}\hat{v}_2\hat{u}^{2k+2}.$$

We can now read off the x^3-torsion in the associated graded object from this for Theorem 3.3. After our d_3, we are left with E_4 as stated in the theorem. All elements are in degrees divisible by 8, and the differentials d_4, d_5, and d_6 have degrees 5, 6, and 7, mod 8, so are all zero.

Our 3 generators are known to be cycles in this filtration, so the differential d_7 is determined by what happens on the coefficients,

$$d_7(v_2^4) = \hat{v}_2 v_2^{-8} = v_2^{-16} = \hat{v}_2^2.$$

Our x^7-torsion is as in Theorem 3.3 and $E_8 = 0$. ∎

Observe that if $q = 1$, we have x^7-torsion generators $\hat{u}^{1-3}$ and as q goes off to infinity, we are just left with our $\hat{u}^2$ from $\mathbb{CP}^\infty$.

Proof of Theorem 1.5 We have already proved (i) and (ii). Recall that $ER(2)^{8*}$ is just $ER(2)^*$ without x or α_1 and α_3, as in Fact 2.1. These coefficients inject into the coefficients $E(2)^{8*}$. To show that $ER(2)^{8*}(B\mathbb{Z}/(2^q)) \to E(2)^{8*}(B\mathbb{Z}/(2^q))$ injects, all we need to do is read off the answer from Theorem 3.3. All elements in the kernel must have an x. The x^1-torsion never has an x non-zero. All of the generators of x^3 and x^7-torsion are in degree zero mod (8) and the degree of x mod (8) is -1. The injection follows.

By the injection we know that z^2 must lie in the description given by Theorem 1.5(ii). In principle, we know all about $E(2)^*(B\mathbb{Z}/(2^q))$. We know that z goes to $[2^{q-1}](\hat{u})$ and $\hat{p}_1$ goes to $\hat{u}c(\hat{u})$. We also know that $0 = [2^q](\hat{u})$. The formal group law gives the series for $c(\hat{u})$ by $0 = \hat{F}(\hat{u}, c(\hat{u}))$.

We need an algorithm that allows us to compute $2z$, z^2, and $2^q\hat{p}_1$ in terms of $z\hat{p}_1^i$ and $\hat{p}_1^i$. We need to be more specific. We want to write our elements in terms of a series using sums of elements $az\hat{p}_1^i$ and $b\hat{p}_1^i$ with $a, b \in \mathbb{Z}_{(2)}[\hat{v}_1, \hat{v}_2]$ and where 2 does not divide a and 2^q does not divide b. From the injection, we know this is possible and our algorithm works for all three cases. We use the same names, z and $\hat{p}_1$, for the images in $E(2)^*(B\mathbb{Z}/(2^q))$.

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Let

$$z = [2^{q-1}](\hat{u}) = f(\hat{u}) = \sum_{i\geq 0} f_i \hat{u}^{i+1} \quad \text{with } f_0 = 2^{q-1},$$
$$[2^q](\hat{u}) = g(\hat{u}) = \sum_{i\geq 0} g_i \hat{u}^{i+1} \quad \text{with } g_0 = 2^q,$$
$$c(\hat{u}) = h(\hat{u}) = \sum_{i\geq 0} h_i \hat{u}^{i+1} \quad \text{with } h_0 = -1.$$

We have

$$f_i, g_i, h_i \in \mathbb{Z}_{(2)}[\hat{v}_1, \hat{v}_2] \subset E(2)^*$$

and

$$2^{q-1}\hat{u} = z - \sum_{i>0} f_i \hat{u}^{i+1}, \quad 2^q \hat{u} = -\sum_{i>0} g_i \hat{u}^{i+1}, \quad -\hat{u} = c(\hat{u}) - \sum_{i>0} h_i \hat{u}^{i+1},$$
$$\hat{u}^2 = -\hat{u}(-\hat{u}) = -\hat{u}\Big(c(\hat{u}) - \sum_{i>0} h_i \hat{u}^{i+1}\Big) = -\hat{p}_1 + \sum_{i>0} h_i \hat{u}^{i+2}.$$

The important facts here are that, mod higher powers of $\hat{u}$, $z = 2^{q-1}\hat{u}$, $0 = 2^q\hat{u}$, and $\hat{p}_1 = -\hat{u}^2$.

We can now present our algorithm for computing $2z$, z^2, and $2^q\hat{p}_1$ using induction. To start, the only coefficient of $\hat{u}$ we can have from our elements of interest is $2^{q-1}\hat{u}$ from z and the injection, which we can replace with the formula above to get z modulo higher powers of $\hat{u}$.

Let us assume that we have succeeded for powers of $\hat{u}$ below $\hat{u}^j$. If we have a term here, $a\hat{u}^j$ with 2^q dividing a, then we rewrite as

$$a\hat{u}^j = (a/2^q)\hat{u}^{j-1}\Big(-\sum_{i>0} g_i \hat{u}^{i+1}\Big) = 0 \mod \hat{u}^{j+1}.$$

If 2^q does not divide a, we have two cases. If j is even, say $j = 2k$, then

$$a\hat{u}^{2k} = a\Big(-\hat{p}_1 + \sum_{i>0} h_i \hat{u}^{i+2}\Big)^k = a(-\hat{p}_1)^k \mod \hat{u}^{j+1}.$$

If j is odd, say $j = 2k+1$, we must use injectivity to see that 2^{q-1} divides a, but not 2^q (as we have already dealt with that). Now we can set

$$a\hat{u}^{2k+1} = (a/2^{q-1})\Big(z - \sum_{i>0} f_i \hat{u}^{i+1}\Big)\Big(-\hat{p}_1 + \sum_{i>0} h_i \hat{u}^{i+2}\Big)^k$$
$$= (a/2^{q-1})z(-\hat{p}_1)^k \mod \hat{u}^{j+1}.$$

This concludes the inductive step. ∎

Remark 5.2 We can do more than this. We can clearly push the elements $2^{q-1}\alpha_i\hat{p}_1$ and $\alpha_i z$ into $E(2)^*(B\mathbb{Z}/(2^q))$ and do the same thing as above to get relations. For example, $\alpha_3 z$ reduces to a series with lead term $2v_2^6 2^{q-1}\hat{u}$, and so we have $2^q\hat{u}$, which lives in higher filtrations. We have not given nice names to elements that live in degree $4 \mod (8)$, so they are not so easy to describe. From Theorem 3.3, we can read off that all x^i-torsion generators in degrees $4*$ inject by the reduction to $E(2)^*(B\mathbb{Z}/(2^q))$. However, in the case of elements in degree $4*$, there are 3 elements (setting $\hat{v}_2 = 1$) in the kernel, namely $\{x^4 z, x^4\hat{p}_1, x^4 z\hat{p}_1\}$. At first glance it appears that we cannot solve

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for these relations completely in $E(2)^*(B\mathbb{Z}/(2^q))$ because of the kernel. However, we know we have injection for degrees $8*$. The only new relations not in degree $8*$ and not divisible by x are for $2^{q-1}\alpha_{\{1,3\}}\hat{p}_1$ and $\alpha_{\{1,3\}}z$. The degrees of these four elements, mod (48), are 4, 28, 20, and 44. The degrees of the 3 elements above that are divisible by x are 12, 44, and 28, so we could possibly have a problem here. For example, write the above $\alpha_3 z = y + ax^4\hat{p}_1$ where no term in the series y is divisible by x. We would like to show that a must be zero. Since α_3 is x^1-torsion, and $x^4\hat{p}_1$ is x^3-torsion, if $a \neq 0$, then y must be an x^3-torsion generator. However, a quick look at Theorem 3.3 shows that there are no x^3-torsion generators in degree equal to 4 mod (8). Only one other element could have a similar problem but it is solved in the same way. The bottom line is that all the relations that involve the α_i can be solved in $E(2)^*(B\mathbb{Z}/(2^q))$ without any involvement of elements divisible by x.

6 The Atiyah-Hirzebruch Spectral Sequence for $ER(2)^*(\mathbb{CP}^\infty)$

There are only 3 differentials in the AHSS for $ER(2)^*(\mathbb{CP}^\infty)$: d_2, d_4, and d_6. To simplify our computations here, we grade over $\mathbb{Z}/(48)$ by setting $\hat{v}_2 = v_2^{-8} = 1$, without any loss of information.

Our goal is to use our results from the BSS to compute the AHSS and then to identify all our terms from the BSS in the AHSS. This immediately gives the AHSS for $ER(2)^*(\mathbb{CP}^n)$, and then we can use it to go back and compute the BSS for $\mathbb{CP}^n$. This is a novel circle of arguments and not necessarily the most efficient approach to $ER(2)^*(\mathbb{CP}^n)$, but the AHSS result is of some interest in its own right. In particular, it is from the AHSS that we get the results on the extra powers of $\hat{p}_1$ for Theorem 1.2.

In general, when we talk about the AHSS, we will use α for $\hat{v}_1$ and u instead of $\hat{u} = v_2^3 u$, but stick with $\hat{v}_1$ and $\hat{u}$ with the BSS.

The degree of u is 2, and it is the natural choice for the AHSS, since

$$H^*(\mathbb{CP}^\infty;\mathbb{Z}_{(2)}) = \mathbb{Z}_{(2)}[u] \quad \text{and} \quad H^*(\mathbb{CP}^\infty;\mathbb{Z}/(2)) = \mathbb{Z}/(2)[u].$$

We describe the AHSS in a sequence of theorems, keeping in mind that we have set $\hat{v}_2 = 1$. To help keep track of degrees, it might be helpful to the reader to remember the table for $ER(2)^*$ in the Appendix A.

Theorem 6.1 *The E_2 term of the (reduced) AHSS for $ER(2)^*(\mathbb{CP}^\infty)$ is*

$$\mathbb{Z}_{(2)}[\alpha,u]\{u, wu, \alpha_1 u, \alpha_2 u, \alpha_3 u\} \quad \textit{with} \quad 2wu = \alpha\alpha_2 u,$$
$$\mathbb{Z}/(2)[\alpha,u]\{x^{1-2}u, x^{1-2}wu\} \qquad \mathbb{Z}/(2)[u]\{x^{3-6}u\}.$$

The differential, d_2, is determined by the multiplicative structure and $d_2(u) = x\alpha u^2$.

$E_3 = E_4$ is

$$\mathbb{Z}_{(2)}[\alpha,u^2]\{\alpha_i u\}, \qquad \mathbb{Z}/(2)[\alpha,u^2]\{x^2\alpha u, x^2 wu\}, \quad 0 \le i < 4,$$
$$\mathbb{Z}_{(2)}[\alpha,u^2]\{u^2, wu^2, \alpha_1 u^2, \alpha_2 u^2, \alpha_3 u^2\} \text{ with } 2wu^2 = \alpha\alpha_2 u^2,$$
$$\mathbb{Z}/(2)[u^2]\{x^{1-6}u^2, x^{1-2}wu^2, x^{2-6}u\}.$$

The reader is spared the complete description of $E_5 = E_6$. We can do this because we know $ER(2)^*(\mathbb{CP}^\infty)$ already. We just identify $ER(2)^*(\mathbb{CP}^\infty)$ in the AHSS, and

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we can ignore those elements, because they cannot be either source or target for differentials. We will not identify all elements just yet though.

Theorem 6.2 *In E_4 of the AHSS for $ER(2)^*(\mathbb{CP}^\infty)$ we identify the x^1-torsion generators from the BSS on the left with the AHSS elements on the right:*

$$\mathbb{Z}_{(2)}[\hat{v}_1, v_2^{\pm 2}, \hat{u}^2]\{2v_2\hat{u}, 2\hat{u}^2\} = \mathbb{Z}_{(2)}[\alpha, u^2]\{\alpha_i u, \alpha_i u^2\} \qquad 0 \le i < 4,$$
$$\mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 4}, \hat{u}^4]\{\hat{v}_1 v_2^2\hat{u}^2, \hat{v}_1\hat{u}^4\} = \mathbb{Z}/(2)[\alpha, u^2]\{\alpha u^2, wu^2\},$$
$$\mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 4}, \hat{u}^4]\{\hat{v}_1\hat{u}^2, \hat{v}_1 v_2^2\hat{u}^4\} = \mathbb{Z}/(2)[\alpha, u^2]\{x^2\alpha u, x^2 wu\}.$$

Theorem 6.3 *After removing the x^1-torsion from E_4 for the AHSS for $ER(2)^*(\mathbb{CP}^\infty)$, what remains are*

$$\mathbb{Z}/(2)[u^2]\{x^{0-6}u^2\}, \qquad \mathbb{Z}/(2)[u^2]\{x^{1-2}wu^2, x^{2-6}u\}.$$

The differential, d_4, is determined by the multiplicative structure and $d_4(u^2) = x^3u^4$.

$E_5 = E_6$ is, after removing all of the x^1-torsion

$$\mathbb{Z}/(2)[u^2]\{x^{1-2}wu^2\}, \qquad \mathbb{Z}/(2)\{x^{2-6}u\}, \qquad \mathbb{Z}/(2)[u^4]\{x^{4-6}u^2\},$$
$$\mathbb{Z}/(2)[u^4]\{x^{0-2}u^4\}, \qquad \mathbb{Z}/(2)[u^4]\{x^{4-6}u^3\}, \qquad \mathbb{Z}/(2)[u^4]\{x^{2-4}u^5\}.$$

We can now describe, and eliminate, the x^3-torsion elements from E_6 before we compute d_6.

Theorem 6.4 *In E_6 of the AHSS for $ER(2)^*(\mathbb{CP}^\infty)$ we identify the elements involved with x^3-torsion from the BSS on the left with the AHSS elements on the right.*

$$\mathbb{Z}/(2)[\hat{u}^8]\{x^{0-2}\hat{u}^8, x^{0-2}v_2^4\hat{u}^4\} = \mathbb{Z}/(2)[u^4]\{x^{0-2}u^4\},$$
$$\mathbb{Z}/(2)[\hat{u}^8]\{x^{0-2}\hat{u}^{10}, x^{0-2}v_2^4\hat{u}^6\} = \mathbb{Z}/(2)[u^4]\{x^{2-4}u^5\},$$
$$\mathbb{Z}/(2)[\hat{u}^8]\{x^{0-2}\hat{u}^4, x^{0-2}v_2^4\hat{u}^8\} = \mathbb{Z}/(2)[u^4]\{x^{4-6}u^2\},$$
$$\mathbb{Z}/(2)[\hat{u}^8]\{\hat{u}^6, v_2^4\hat{u}^{10}\} = \mathbb{Z}/(2)[u^4]\{x^6u^3\},$$
$$\mathbb{Z}/(2)[\hat{u}^8]\{x^{1-2}\hat{u}^6, x^{1-2}v_2^4\hat{u}^{10}\} = \mathbb{Z}/(2)[u^4]\{x^{1-2}wu^4\}.$$

Theorem 6.5 *After removing the x and x^3-torsion from E_6 for the AHSS for $ER(2)^*(\mathbb{CP}^\infty)$, what remains are*

$$\mathbb{Z}/(2)[u^4]\{x^{1-2}wu^2\}, \qquad \mathbb{Z}/(2)\{x^{2-6}u\}, \qquad \mathbb{Z}/(2)[u^4]\{x^{4-5}u^3\}$$

The differential, d_6, is determined by the multiplicative structure and $d_6(x^4u^3) = xwu^6$.

$E_7 = E_\infty$ is, after removing all of the x^1 and x^3 torsion:

$$\mathbb{Z}/(2)\{x^{2-6}u\} \qquad \mathbb{Z}/(2)\{x^{1-2}wu^2\}.$$

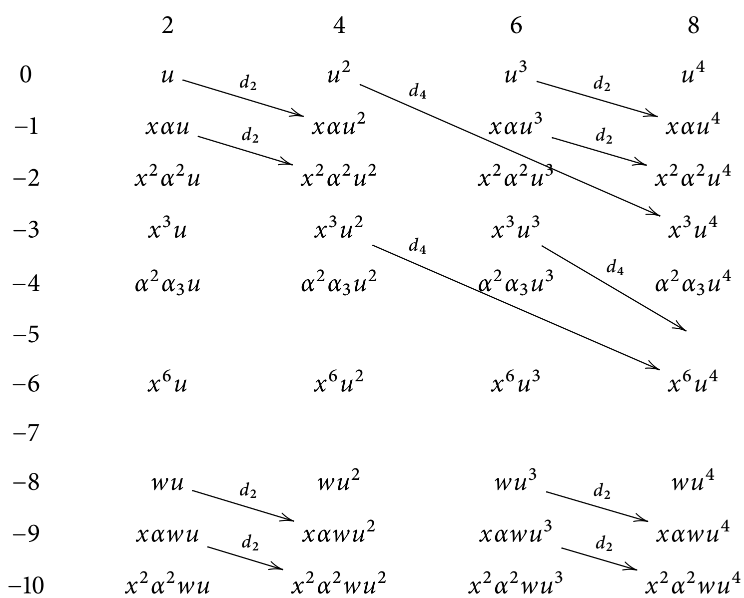
Theorem 6.6 *In E_∞ of the AHSS for $ER(2)^*(\mathbb{CP}^\infty)$ we identify the elements involved with x^7-torsion from the BSS on the left with the AHSS elements on the right:*

$$\mathbb{Z}/(2)\{x^{0-4}\hat{u}^2\} = \mathbb{Z}/(2)\{x^{2-6}u\} \qquad \mathbb{Z}/(2)\{x^{5-6}\hat{u}^2\} = \mathbb{Z}/(2)\{x^{1-2}wu^2\}.$$

Theorem 6.7 *Let $\hat{p}_1$ be the element constructed in $ER(2)^*(\mathbb{CP}^\infty)$ that maps to $\hat{u}c(\hat{u})$, which in the associated graded object is $\hat{p}_1 = -\hat{u}^2$. Then, in the AHSS, the elements are represented as follows:*

$$\hat{p}_1 = x^2u, \qquad \hat{p}_1^2 = x^4u^2, \qquad \hat{p}_1^3 = x^6u^3, \qquad \hat{p}_1^4 = u^8,$$
$$\hat{p}_1^{4k+1} = x^2u^{8k+1}, \quad \hat{p}_1^{4k+2} = x^4u^{8k+2}, \quad \hat{p}_1^{4k+3} = x^6u^{8k+3}, \quad \hat{p}_1^{4k+4} = u^{8k+8}.$$

The rest of this section consists of the proofs for all of the theorems stated in this section. Before we embark on that trip, we offer a small visual guide at the request of the referee. Keep in mind that we have made our theory $ER(2)^*(-)$ 48-periodic, and we only offer degrees zero through minus 10 for the coefficients here. We also truncate as if this was $\mathbb{CP}^4$. Also, many terms have arbitrarily high powers of α on them. If a term has an x in it, it represents a $\mathbb{Z}/(2)$. If not, it represents a $\mathbb{Z}_{(2)}$. The differential, d_6, is not in the range of our picture. Neither is $\hat{p}_1$ or $\hat{p}_1^2$, but $\hat{p}_1^3$ is represented here as x^6u^3, in filtration 6 as opposed to the expected filtration 12. Note also that the d_4 on x^3u^3 goes off our scale. That means that if we were really computing $\mathbb{CP}^4$, the element x^3u^3 would have to be a permanent cycle. As it stands, all of the elements in the part of the AHSS pictured below left after d_2 and d_4 are permanent cycles.



Proofs of all the theorems We take a short side trip to think about the BSS for $\mathbb{CP}^2$. It starts with $E(2)^*$ free on $\hat{u}$ and $\hat{u}^2$. The computation of d_1 is identical to that for $\mathbb{CP}^\infty$ by naturality, and we have that $d_1(v_2\hat{u})$ is non-zero, but $v_2\hat{u} = v_2(v_2^3u) = v_2^4u$ and d_1 commutes with v_2^2, so d_1 is non-zero on u, so u does not exist in E_∞ for the AHSS.

In the AHSS for $\mathbb{CP}^2$, the only way u can go away is if $d_2(u) = x\alpha u^2$, since this is the only element in the degree of the image. Technically, we could have $d_2(u) =$

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$x\alpha^{3k+1}u^2$, but this would immediately conflict with the answer we get from the BSS. So, $d_2(u) = x\alpha u^2$ is what must happen, and by naturality, this happens in the AHSS for $\mathbb{CP}^\infty$ as well.

Because $2x = 0$, $d_2(u^2) = 0$. All we need to do to compute E_3 is find the kernel of d_2 on u^{2k+1} and the cokernel on u^{2k}. We know that $x\alpha_i = 0$, where we include $\alpha_0 = 2$. We also know that $x^3\alpha = 0$. The computation of E_3 is straightforward and as stated.

For degree reasons, there is no d_3 (or any d_r, r odd), so we have $E_4 = E_3$.

Before we compute our d_4, we want to identify all of the x-torsion generators and eliminate them from consideration, that is, make the identifications in Theorem 6.2. We begin with the first term of the x-torsion from Theorem 3.1. Making the substitution $\hat{u} = v_2^3 u$, and working over $\mathbb{Z}_{(2)}[v_2^{\pm 2}]$, this turns into $\mathbb{Z}_{(2)}[\hat{v}_1, v_2^{\pm 2}, u^2]\{2u, 2u^2\}$. Making the identification, $2v_2^{2i} = \alpha_i$, we find we have the first line of Theorem 6.2.

Now break up the second term of the x-torsion in Theorem 3.1 into the parts remaining on the left of Theorem 6.2. With the usual substitutions $\hat{u} = v_2^3 u$ and $w = \hat{v}_1 v_2^4$, the second line of Theorem 6.2 becomes obvious, although it must be kept in mind that the generators do not correspond, but it is the whole module that is the same.

The final line of Theorem 6.2 is somewhat more of a challenge. Here we recall, looking at the image in $E(2)^*$, that v_2^2 and v_2^6 are $\frac{1}{2}\alpha_1$ and $\frac{1}{2}\alpha_3$, respectively. To find a representative for $\alpha\frac{1}{2}\alpha_1 u^2$ and $\alpha\frac{1}{2}\alpha_3 u^2$, the only elements available are x^2wu and $x^2\alpha u$, respectively. This, together with our usual substitution of $\hat{u} = v_2^3 u$, is enough to give us our last line in the identification.

After taking all of the x-torsion from the BSS out of E_4 of the AHSS, we get the stated remaining terms in E_4.

Recall that $\hat{u} = v_2^3 u$ so $u^4 = \hat{u}^4 v_2^{-12}$, and this is in the image of d_3 in the BSS, so it is both a permanent cycle and x^3-torsion. So, x^3u^4 must be the target of a differential. The differential d_2 did not hit it. The only possible d_4 is $d_4(u^2) = x^3u^4$. In principle, a d_6 could hit it, but this would require $d_6(z_2 u) = x^3u^4$ for some element z_2 with the degree of z_2 equal to 2, and there is no such element. We conclude that we must have $d_4(u^2) = x^3u^4$.

Other than u^2, we have one other generator (over $ER(2)^*$) in what remains of E_4 that d_4 could be non-zero on. It is x^2u.

The element x^2u is the most interesting. We know that $\hat{u}c(\hat{u})$ exists. In the AHSS, this would be $-u^2v_2^6 = -\frac{1}{2}\alpha_3 u^2$. This does not exist in the filtration associated with u^2 in the AHSS, but $\alpha_3 u^2$ does. In order to divide it by 2, we have to go to the previous filtration, where we find x^2u, which must represent $\hat{u}c(\hat{u})$, and so cannot have a differential on it.

Now we are free to compute d_4 from $d_4(u^2) = x^3u^4$. We get the obvious

$$d_4(x^{0-3}u^{4k+2}) = x^{3-6}u^{4k+4}.$$

In addition, we know that multiplication by x^2u commutes with d_4, so we get another family:

$$d_4(x^{2-3}u^{4k+3}) = x^{5-6}u^{4k+5}.$$

There is so little left of E_4 without the x-torsion that d_4 is easy to compute and gives the stated result for $E_5 = E_6$ without the x-torsion.

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Before we continue on to d_6, we want to identify the x^3-torsion from the BSS computation. Rewrite the x^3-torsion from the BSS of Theorem 3.1 into the terms on the left of Theorem 6.4.

After that, make the substitution $\hat{u} = v_2^3 u$, keeping in mind our ever present $v_2^8 = 1$. With this, we have $\hat{u}^2 = v_2^6 u^2$, $\hat{u}^4 = v_2^4 u^4$, $\hat{u}^6 = v_2^2 u^6$, and $\hat{u}^8 = u^8$. The terms on the left of Theorem 6.4 reduce to, in order,

$$\mathbb{Z}/(2)[u^4]\{x^{0-2}u^4, x^{0-2}v_2^6u^6, x^{0-2}v_2^4u^4, x^{0-2}v_2^2u^6\}.$$

The one easy case now is the first one, and it gives the first line of Theorem 6.4.

For the second line, we look at $v_2^6u^6 = \frac{1}{2}\alpha_3 u^6$. This must be some element $z_i u^i$ in our E_6, with $0 < i < 6$, and our z_i must have degree $-24 - 2i$. The only possibility is x^2u^5. This takes care of the second line.

Next we need to consider $v_2^4u^4 = \frac{1}{2}\alpha_2 u^4$. For this element, we want some $z_i u^i$, with $0 < i < 4$, and the degree of z_i to be $-16 - 2i$. The only possible element here is x^4u^2. This completes the third line.

The fourth line is for $v_2^2u^6 = \frac{1}{2}\alpha_1 u^6$. This must be some element $z_i u^i$, with $0 < i < 6$, and the degree of z_i equal to $-2i$. The only element that meets this criteria is x^6u^3, so this must be it. Unlike the others, this one does not have x^{1-2} times it non-zero in the AHSS. So, we are missing the fifth line.

The unfortunate consequence of this is that we can not remove all the x^3-torsion from E_6 before we compute d_6, but we have to leave the

$$\mathbb{Z}/(2)[u^4]\{x^{1-2}wu^4\}$$

in E_6. So, after taking out all x-torsion and most of the x^3-torsion, all we have left in E_6 is

$$\mathbb{Z}/(2)[u^4]\{x^{1-2}wu^2\}, \qquad \mathbb{Z}/(2)\{x^{2-6}u\},$$
$$\mathbb{Z}/(2)[u^4]\{x^{4-5}u^3\}, \qquad \mathbb{Z}/(2)[u^4]\{x^{1-2}wu^4\}.$$

Now comes a tricky part. We have identified our x^7-torsion generator, $\hat{p}_1 = x^2u$, already. The second term here can be eliminated as $x^{0-4}\hat{p}_1$. We are now missing 3 bits of information. We do not know $x^{5-6}(x^2u)$, $x^{1-2}(x^6u^3)$, and any differentials that we might have.

Either we have $x^5(x^2u) = xwu^2$, or there must be a differential on xwu^2, *i.e.*, $d_{2i}(xwu^2) = zu^{2+i}$ where the degree of z must be the degree of xw minus $2i - 1$ (mod 48). There is no such z left in our E_6. Consequently, we have $x^5(x^2u) = xwu^2$.

Next, either $x(x^6u^3) = xwu^4$, or there must be a differential on the last term. A similar argument to the above shows there is no such element. With these extension problems solved, all that is left of E_6 is

$$\mathbb{Z}/(2)[u^4]\{x^{1-2}wu^6\}, \qquad \mathbb{Z}/(2)[u^4]\{x^{4-5}u^3\}.$$

The only way they can go away is to have $d_6(x^4u^3) = xwu^6$.

Theorem 6.7 is now just a matter of inspection.

This completes the proof of all the theorems in this section. ∎

Remark 6.8 We don't need this, but it is an interesting observation, so we comment on it. We just showed that in the AHSS, the solutions to some extension problems are

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now clear. We have

$$2(x^2u) = \alpha_3 u^2, \qquad 2(x^2u)^2 = 2x^4u^2 = \alpha_2 u^4,$$
$$2(x^2u)^3 = 2x^6u^3 = \alpha_1 u^6.$$

Although the powers of $\hat{p}_1$, represented by x^2u, are not in the expected filtrations, 2 times them are.

7 $ER(2)^*(\mathbb{CP}^n)$

We can first note that we have already computed the AHSS for $ER(2)^*(\mathbb{CP}^n)$, because the E_2 term from the AHSS for $\mathbb{CP}^\infty$ surjects. That means that the only differences are given by the differential d_2 on the u^n filtration, d_4 on the $u^{n,n-1}$ filtrations, and d_6 on the $u^{n,n-1,n-2}$ filtrations. Because there are sometimes no targets, elements survive that are not in the image from $ER(2)^*(\mathbb{CP}^\infty)$. We will use these facts to compute the BSS for $ER(2)^*(\mathbb{CP}^n)$. As usual for our evenly graded BSS's, we have $d_{2,4,6} = 0$.

To be able to combine calculations we need a notational convention: $a^{\{b,c\}}d^{\{e,f\}}$ means $a^b d^e$ and $a^c d^f$. We have stated the description of $ER(2)^*(\mathbb{CP}^n)$ in Section 3 and Theorem 1.2. Here we describe the BSS and prove everything. Our computations are complicated due to having 8 distinct cases.

Theorem 7.1 *Filtering $E(2)^*(\mathbb{CP}^{2j}) = E(2)^*[\hat{u}]/(\hat{u}^{2j+1})$ by powers of $\hat{u}$, we give E_r as an associated graded object of the actual E_r of the BSS for the reduced $ER(2)^*(\mathbb{CP}^{2j})$:*

$$E_1 = E(2)^*[\hat{u}]/(\hat{u}^{2j})\{\hat{u}\},$$
$$E_2 = E_3 = \mathbb{Z}/(2)[v_2^{\pm 2}, \hat{u}^2]/(\hat{u}^{2j})\{\hat{u}^2\},$$
$$E_4 = E_5 = E_6 = E_7 = \mathbb{Z}/(2)[v_2^{\pm 4}]\{\hat{u}^2, v_2^2\hat{u}^{2j}\}.$$

Theorem 7.2 *Filtering $E(2)^*(\mathbb{CP}^{2j+1}) = E(2)^*[\hat{u}]/(\hat{u}^{2j+2})$ by powers of $\hat{u}$, we give E_r as an associated graded object of the actual E_r of the BSS for the reduced $ER(2)^*(\mathbb{CP}^{2j+1})$.*

$$E_1 = E(2)^*[\hat{u}]/(\hat{u}^{2j+1})\{\hat{u}\}$$
$$E_2 = E_3 = \mathbb{Z}/(2)[v_2^{\pm 2}, \hat{u}^2]/(\hat{u}^{2j})\{\hat{u}^2\} \quad \mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 2}]\{v_2\hat{u}^{2j+1}\}$$
$$E_4 = E_5 = \mathbb{Z}/(2)[v_2^{\pm 4}]\{\hat{u}^2, v_2^2\hat{u}^{2j}, v_2^{2j+1}\hat{u}^{2j+1}\}$$
$$E_5 = E_6 = E_7 \quad \textit{for } j = 4k+1 \textit{ and } 4k+3$$
$$E_6 = E_7 = \mathbb{Z}/(2)[v_2^{\pm 4}]\{\hat{u}^2\} \quad \textit{for } j = 4k \textit{ and } 4k+2.$$

The rest of this section is dedicated to the computations proving the various theorems about $ER(2)^*(\mathbb{CP}^n)$. Up to E_5, the computations pretty much follow from our work with $\mathbb{C}P^\infty$. After that, technically there are 8 cases. We present a reference guide (at the request of the referee) for d_5 and d_7 in these 8 cases. We will ignore the $\hat{u}^2$ and $v_2^4\hat{u}^2$ terms because the d_7 there comes from $\mathbb{C}P^\infty$. We can get all 8 cases from just 3

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diagrams:

$$\mathbb{CP}^{4k+1}: \quad \begin{array}{c} v_2^2\hat{u}^{4k} \xrightarrow{d_5} v_2\hat{u}^{4k+1} \\[2em] v_2^6\hat{u}^{4k} \xrightarrow{d_5} v_2^5\hat{u}^{4k+1}, \end{array}$$

$$\mathbb{CP}^{8k+3}: \quad \begin{array}{ccc} v_2^2\hat{u}^{8k+2} & & v_2^3\hat{u}^{8k+3} \\ d_7 \big\uparrow & & \big\downarrow d_7 \\ v_2^6\hat{u}^{8k+2} & & v_2^7\hat{u}^{8k+3}, \end{array}$$

$$\mathbb{CP}^{8k+7}: \quad \begin{array}{ccc} v_2^2\hat{u}^{8k+6} & & v_2^3\hat{u}^{8k+7} \\ \big\downarrow d_7 & & d_7 \big\uparrow \\ v_2^6\hat{u}^{8k+2} & & v_2^7\hat{u}^{8k+7}. \end{array}$$

This displays the results for the odd spaces, but we can restrict $8k+3$ and $8k+7$ to $8k+2$ and $8k+6$, respectively to get the appropriate d_7 for those spaces. Then $8k$ and $8k+6$ are the same and $8k+2$ and $8k+4$ are the same. This should provide a guide for when the proofs get confusing.

Proofs For even spaces, our BSS E_1 is $E(2)^*(\mathbb{CP}^{2j}) = E(2)^*[\hat{u}]/(\hat{u}^{2j+1})$. Since $E(2)^*(\mathbb{CP}^\infty)$ surjects to this, we inherit our computation of $d_{1,1}$, $d_{1,2}$, and consequently, the entirety of d_1. The x-torsion and E_2 are as stated.

We also inherit d_3 from $\mathbb{CP}^\infty$ and can read off the x^3-torsion and E_4 directly. Note that at this stage, the only difference between $\mathbb{CP}^{2j}$ and $\mathbb{CP}^\infty$ is the lack of a d_3 on $v_2^2\hat{u}^{2j}$.

For degree reasons, we can only have d_7, not d_5, so $E_4 = E_7$. On the first term, d_7 is inherited from $\mathbb{CP}^\infty$, but on the last term, there are several cases.

We consider the map:

$$\mathbb{CP}^{8k+2i} \longrightarrow S^{16k+4i}.$$

In $E(2)^*(-)$, the generator for the sphere, s_{16k+4i} maps to u^{8k+2i}. The generator for the sphere must be a permanent cycle, so must $u^{8k+2i} \in E(2)^*(\mathbb{CP}^{8k+2i})$. We have $\hat{u} = v_2^3 u$, so we know that the following is a permanent cycle:

$$u^{8k+2i} = (v_2^{-3}\hat{u})^{8k+2i} = v_2^{-24k-6i}\hat{u}^{8k+2i} = v_2^{-6i}\hat{u}^{8k+2i} = v_2^{2i}\hat{u}^{8k+2i}.$$

Unfortunately, this only exists in E_7 for $i = \{1,3\}$. So we have

$$d_7\big(v_2^4(v_2^{2i}\hat{u}^{8k+2i})\big) = v_2^{2i}\hat{u}^{8k+2i} \qquad i = \{1,3\}.$$

This gives the x^7-torsion as stated for $8k+2$ and $8k+6$.

We now have to deal with $8k$ and $8k+4$ where we cannot use the sphere and naturality. We will try to do both at the same time even though they have different outcomes. We have two elements in each case: $v_2^{\{2,6\}}\hat{u}^{8k}$ and $v_2^{\{2,6\}}\hat{u}^{8k+4}$.

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In each case, one must be the source and one the target. If an element is a target, it has to exist in the AHSS, and would be represented by one of

$$v_2^{\{2,6\}}\hat{u}^{8k} = v_2^{\{2,6\}}u^{8k} \quad \text{and} \quad v_2^{\{2,6\}}\hat{u}^{8k+4} = v_2^{\{6,2\}}u^{8k+4}.$$

It is important to note how in the $8k+4$ case, the powers of v_2 got switched around in the transition from $\hat{u}$ to u.

The element we are searching for (the target) cannot be in the image from $\mathbb{CP}^\infty$ because there is no x^7-torsion element there that could come and hit it. These elements cannot be represented in the filtration occupied by $u^{8k+\{0,4\}}$ because

$$v_2^2 u^{8k+\{0,4\}} = \frac{1}{2}\alpha_1 u^{8k+\{0,4\}} \quad \text{and} \quad v_2^6 u^{8k+\{0,4\}} = \frac{1}{2}\alpha_3 u^{8k+\{0,4\}}$$

do not exist there. Because they cannot be in the image from $\mathbb{CP}^\infty$, they must be a source for an AHSS differential in $\mathbb{CP}^\infty$ but not in $\mathbb{CP}^n$. So the class we search for must be in the filtration of $u^{8k+\{-1,3\}}$. (There is no d_6 on the filtration for $u^{8k+\{-2,2\}}$, and so everything in that filtration is in the image from $\mathbb{CP}^\infty$ and not what we are looking for.) The v_2^2 case would have to be $z_1 u^{8k+\{-1,3\}}$ with degree of z_1 equal to $-10 = 38$, and the v_2^6 case would have to be $z_2 u^{8k+\{-1,3\}}$ with degree of z_2 equal to $-34 = 14$. The v_2^2 candidate would have to be $z_1 = x^2 w\alpha^{3i+2}$, but we have already shown that this is a permanent cycle representing an element. On the other hand, a good choice for z_2 is x^2. Since we know that for $\mathbb{CP}^\infty$ there is an AHSS d_4 on $x^2 u^{8k+\{-1,3\}}$, this must be our choice. We could track down the x^7-torsion elements in the AHSS, but this is not necessary. We have, in the BSS,

$$d_7(v_2^2\hat{u}^{8k}) = v_2^6\hat{u}^{8k} \qquad d_7(v_2^6\hat{u}^{8k+4}) = v_2^2\hat{u}^{8k+4},$$

and our x^7 torsion is generated as stated.

For odd n, our BSS E_1 is $E(2)^*(\mathbb{CP}^{2j+1}) = E(2)^*[\hat{u}]/(\hat{u}^{2j+2})$. Since $E(2)^*(\mathbb{CP}^\infty)$ surjects to this, we inherit our computation of $d_{1,1}$, $d_{1,2}$, and, consequently, the entirety of d_1. We can read off the x-torsion and E_2 as stated.

We also inherit d_3 on the first part of the stated E_3 and can read off the x^3-torsion it gives as well as its contribution to E_4. The last part of E_3 is different. In order to determine this, we use the map

$$\mathbb{CP}^{2j+1} \longrightarrow S^{4j+2}.$$

In $E(2)^*(-)$, the generator for the sphere, s_{4j+2} maps to u^{2j+1}. Since we know the generator for the sphere must be a permanent cycle, so must $u^{2j+1} \in E(2)^*(\mathbb{CP}^{2j+1})$. We have $\hat{u} = v_2^3 u$, so the permanent cycle is

$$u^{2j+1} = (v_2^{-3}\hat{u})^{2j+1} = v_2^{-6j-3}\hat{u}^{2j+1} = v_2^{2j-3}\hat{u}^{2j+1}.$$

These elements are all there in E_3 so we know d_3 by naturality from the sphere, which is just like the coefficients. It is just

$$d_3\big(v_2^2(v_2^{2j-3}\hat{u}^{2j+1})\big) = \hat{v}_1 v_2^{-4} v_2^{2j-3}\hat{u}^{2j+1} = \hat{v}_1 v_2^{2j+1}\hat{u}^{2j+1}.$$

As a result, the x^3-torsion from this part is as stated and likewise for the contribution to E_4.

We only have d_5 and d_7 left. The first term is dealt with by a d_7 from $\mathbb{CP}^\infty$.

We know that $v_2^{2j-3}\hat{u}^{2j+1}$ is a permanent cycle. That means it must be in the image of either d_5 or d_7.

There are two ways to go about determining this. We could compute $ER(2)^*(-)$ for $\mathbb{CP}^{2j+1}/\mathbb{CP}^{2j-1}$ and look at the map of BSSs from this to $\mathbb{CP}^{2j+1}$. This is similar to what the first and third author did with $\mathbb{RP}^{2n}$ in [KW08a]. Or, the way we do it here, is to look at the AHSS to determine which differential it is.

For degree reasons, the only possible candidates for the BSS d_5 are

$$d_5\big(v_2^{\{2,6\}}\hat{u}^{8k+\{0,4\}}\big) = v_2^{\{1,5\}}\hat{u}^{8k+\{1,5\}}$$

and, since d_5 commutes with v_2^4 (from the coefficients),

$$d_5\big(v_2^{\{6,2\}}\hat{u}^{8k+\{0,4\}}\big) = v_2^{\{5,1\}}\hat{u}^{8k+\{1,5\}}.$$

Recall that $v_2^{2j-3}\hat{u}^{2j+1} = u^{2j+1}$ is a BSS permanent cycle, so $v_2^5\hat{u}^{8k+1} = u^{8k+1}$ and $v_2\hat{u}^{8k+5} = u^{8k+5}$ are both BSS and AHSS permanent cycles for $n = 8k+1$ and $n = 8k+5$, respectively. In the AHSS, we know that $d_4(x^2u^{8k+\{-1,3\}}) = x^5u^{8k+\{1,5\}}$. Thus, we must have a BSS d_5, because $u^{8k+\{1,5\}}$ is x^5-torsion. We get, in the BSS, for purely dimensional reasons, that the candidate d_5 above is the correct differential.

Our x^5-torsion is as stated as well as the $E_6 = E_7$. The E_7 inherits d_7 from $\mathbb{CP}^\infty$, so our x^7 torsion is as stated.

We have seen that the only possibilities for d_5 occur, so for $8k + \{3,7\}$, we are left with only d_7. Recall E_7 is:

$$\mathbb{Z}/(2)[v_2^{\pm 4}]\big\{\hat{u}^2, v_2^2\hat{u}^{8k+\{2,6\}}, v_2^{2j+1}\hat{u}^{8k+\{3,7\}}\big\}.$$

The first term is dealt with by a d_7 from $\mathbb{CP}^\infty$. The degrees are not right for d_7 to go from the second term of E_7 to the third, so each term must disappear on its own from a d_7.

The third term is easy because it comes directly from the sphere, so we can read off our x^7-torsion generators from it.

The second term is easy as well, since we can map to the space $\mathbb{CP}^{8k+\{2,6\}}$, where we know the differential on these terms. We can read off the new x^7-torsion from the middle term. And, of course, there is the x^7-torsion from $\mathbb{Z}/(2)\{\hat{u}^2\}$.

All that remains is to prove Theorem 1.2. We can just read off all the elements in degrees $16*$ from our theorems and then use Theorem 6.7 to identify them with powers of $\hat{p}_1$. ∎

8 The Norm

This section is devoted to a proof of Theorem 1.7. The analogous result in [Lor16] for $ER(n)^*(\mathbb{CP}^\infty)$ requires significant theory, mainly because of the arbitrary n. However, in our case, with $n = 2$, we have explicitly written down all of the elements in $ER(2)^*(B\mathbb{Z}/(2^q))$ in Theorem 3.3, so our job here is just a matter of chasing these elements around.

We refer the reader to [Lor16] for all necessary background on the norm, N_*, and its behavior for $\mathbb{CP}^\infty$. In particular, because z is an element of $ER(2)^*(B\mathbb{Z}/(2^q))$, the norm commutes with it. Following [Lor16, Section 7] and Definition 1.6, the reduction of $\operatorname{im}(N_*^{\text{res}})$ to $E(2)^*(B\mathbb{Z}/(2^q))$ is generated by

$$\mathbb{Z}_{(2)}[\hat{v}_1, v_2^{\pm 2}][[\hat{u}c(\hat{u})]]\{\hat{u}+c(\hat{u}), v_2(\hat{u}-c(\hat{u}), z(\hat{u}+c(\hat{u})), v_2 z(\hat{u}-c(\hat{u}))\}.$$

In [Lor16, Lemma 10.3], it is also shown that $\operatorname{im}(N_*^{\text{res}})$ is x-torsion. This is enough background to get us started with our proof.

We outline our proof as it is somewhat technical. From Theorem 3.3, we know the x^1-torsion and $\operatorname{im}(N_*^{\text{res}})$ lies in this x^1-torsion. We first identify this image. After that, we take what is left of Theorem 3.3 and rewrite it in terms of $ER(2)^*$, z, and $\hat{p}_1$. From there it is easy to see a surjective map of

$$ER(2)^*[[\hat{p}_1, z]]/(J) \longrightarrow ER(2)^*\big(B\mathbb{Z}/(2^q)\big)/(\operatorname{im}(N_*^{\text{res}}).$$

Then all that remains is to identify these two.

Proof of Theorem 1.7 We start by computing the associated graded object for

$$ER(2)^*\big(B\mathbb{Z}/(2^q)\big)/\big(\operatorname{im}(N_*^{\text{res}})\big).$$

Recall that we have $ER(2)^*(B\mathbb{Z}/(2^q))$ from Theorem 3.3. The x^1-torsion generators in (the associated graded object for) $ER(2)^*(B\mathbb{Z}/(2^q))$ are

$$\mathbb{Z}/(2^{q-1})[\hat{v}_1, v_2^{\pm 2}, \hat{u}^2]\{2v_2\hat{u}, 2\hat{u}^2\} \quad \mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 2}, \hat{u}^2]\{\hat{v}_1\hat{u}^2, 2^{q-1}\hat{v}_1\hat{u}^3\}.$$

The x^3-torsion generators are

$$\mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 4}]\{2^{q-1}\hat{v}_1\hat{u}\} \qquad \mathbb{Z}/(2)[v_2^{\pm 4}, \hat{u}^2]\{\hat{u}^4, 2^{q-1}\hat{u}^5\}.$$

The x^7-torsion generators are $\mathbb{Z}/(2)[\hat{v}_2^{\pm 1}]\{2^{q-1}\hat{u}, \hat{u}^2, 2^{q-1}\hat{u}^3\}$.

We now remove the elements coming from $\operatorname{im}(N_*^{\text{res}})$. Modulo powers of $\hat{u}^2$, we have that

$$\hat{v}_1^i v_2^{2j}\hat{u}^{2k} v_2\big(\hat{u}-c(\hat{u})\big) \longrightarrow \hat{v}_1^i v_2^{2j}\hat{u}^{2k} 2v_2\hat{u}.$$

This is the very first term of the x^1-torsion above.

Our next concern is

$$\hat{u} \longrightarrow N_*(\hat{u}) = \xi(\hat{p}_1) \longrightarrow \hat{u}+c(\hat{u}).$$

By Lemma 2.7, we have

$$\hat{u}+c(\hat{u}) = \hat{v}_1\hat{u}^2 + \hat{v}_2\hat{u}^4 \qquad \text{mod } (2, \hat{v}_1^2, \hat{u}^5).$$

We have eliminated the first term in our x^1-torsion and now use the first term above to set $\hat{v}_1\hat{u}^2$ and $z\hat{v}_1\hat{u}^2 = 2^{q-1}\hat{v}_1\hat{u}^3$ equal to 0. That reduces our remaining x^1-torsion to only

$$\mathbb{Z}/(2^{q-1})[v_2^{\pm 2}, \hat{u}^2]\{2\hat{u}^2\}.$$

The x^3 and x^7-torsion remain unaffected, since $\operatorname{im}(N_*^{\text{res}})$ is x^1-torsion. We did not use the term $zv_2(\hat{u}-c(\hat{u}))$, but there is nothing left in this degree to hit.

To consolidate, the x^i-torsion generators of

$$ER(2)^*(B\mathbb{Z}/(2^q))/(\operatorname{im}(N_*^{\text{res}}))$$

are described as

$$\begin{array}{ll} x^1\text{-torsion} & \mathbb{Z}/(2^{q-1})[v_2^{\pm 2}, \hat{u}^2]\{2\hat{u}^2\}, \\ x^3\text{-torsion} & \mathbb{Z}/(2)[\hat{v}_1, v_2^{\pm 4}]\{2^{q-1}\hat{v}_1\hat{u}\}, \quad \mathbb{Z}/(2)[v_2^{\pm 4}, \hat{u}^2]\{\hat{u}^4, 2^{q-1}\hat{u}^5\}, \\ x^7\text{-torsion} & \mathbb{Z}/(2)[\hat{v}_2^{\pm 1}]\{2^{q-1}\hat{u}, \hat{u}^2, 2^{q-1}\hat{u}^3\}. \end{array}$$

We set $\hat{v}_2 = 1 = v_2^{-8}$ to simplify our computations.

Having done this, we want to rewrite our answer in terms of the elements this description represents, namely, in our associated graded object: $2v_2^{2i} = \alpha_i$, $\hat{u}^2 = -\hat{p}_1$, $v_2^4\hat{u}^4 = \hat{v}_1 v_2^4 \hat{u}^2 = w\hat{p}_1$, $\hat{v}_1 v_2^4 = w$, and $z = 2^{q-1}\hat{u}$.

Now we rewrite $ER(2)^*(B\mathbb{Z}/(2^q))/(\mathrm{im}(N_*^{\mathrm{res}}))$ as

$$\begin{array}{ll} x^1\text{-torsion} & \mathbb{Z}/(2^{q-1})[\hat{p}_1]\{\alpha_i\hat{p}_1\} \quad 0 \le i < 4, \\ x^3\text{-torsion} & \mathbb{Z}/(2)[\hat{v}_1]\{\hat{v}_1 z, wz\} \quad \mathbb{Z}/(2)[\hat{p}_1]\{\hat{p}_1^2, w\hat{p}_1, z\hat{p}_1^2, zw\hat{p}_1\}, \\ x^7\text{-torsion} & \mathbb{Z}/(2)\{z, \hat{p}_1, z\hat{p}_1\}. \end{array}$$

From this description we see that there is a map

$$ER(2)^*[[\hat{p}_1, z]] \longrightarrow ER(2)^*(B\mathbb{Z}/(2^q))/(\mathrm{im}(N_*^{\mathrm{res}})).$$

Furthermore, this map must map (J) to zero. The above analysis tells us that the unusual elements with the α_i that must be expressed in higher filtrations can be done so, modulo the image of N_*^{res}, in terms of $ER(2)^*$, $\hat{p}_1$, and z. We already knew that this could be done for $2z = \alpha_0 z$, z^2, and $2^q\hat{p}_1$. This allows us to assert that relations, mod image of N_*, exist like this for $\alpha_{\{1,2,3\}}z$ and $2^{q-1}\alpha_{\{1,2,3\}}\hat{p}_1$. These are the relations we use in our Theorem 1.7.

Taken altogether, we get our map, which is now already obviously surjective:

$$\frac{ER(2)^*[[\hat{p}_1, z]]}{(J)} \longrightarrow \frac{ER(2)^*(B\mathbb{Z}/(2^q))}{(\mathrm{im}(N_*^{\mathrm{res}}))}.$$

All that remains is to show that this map is an isomorphism. We do this by proving it is an isomorphism on the associated graded objects. To do that, we analyze the source side of this map.

We begin with $ER(2)^*$ from Fact 2.1. We rewrite it as

$$\begin{array}{ll} x^1\text{-torsion} & \mathbb{Z}[\hat{v}_1]\{\alpha_i\} \quad 0 \le i < 4, \\ x^3\text{-torsion} & \mathbb{Z}/(2)[\hat{v}_1]\{\hat{v}_1, w\}, \\ x^7\text{-torsion} & \mathbb{Z}/(2)\{1\}. \end{array}$$

We continue with our filtration and associated graded object. First, we make everything free over this on $\hat{p}_1^i$ and $\hat{p}_1^i z$. We can make $z^2 = 0$ using our filtration. Likewise, $2^{q-1}\alpha_i\hat{p}_1 = 0 = \alpha_i z$. Writing this down, we will have used every relation except $\xi(\hat{p}_1) = 0$. What we have at this stage is

$$\begin{array}{ll} x^1\text{-torsion} & \mathbb{Z}/(2^{q-1})[\hat{v}_1, \hat{p}_1]\{\alpha_i\hat{p}_1\} \quad 0 \le i < 4, \\ x^3\text{-torsion} & \mathbb{Z}/(2)[\hat{v}_1, \hat{p}_1]\{\hat{v}_1\hat{p}_1, w\hat{p}_1, \hat{v}_1 z, wz\}, \\ x^7\text{-torsion} & \mathbb{Z}/(2)[\hat{p}_1]\{\hat{p}_1, z\}. \end{array}$$

Next we take out the first term of $\xi(\hat{p}_1)$, *i.e.*, $\hat{v}_1\hat{p}_1$. This is a bit different from before when the $\mathrm{im}(N_*^{\mathrm{res}})$ was a submodule of $ER(2)^*(B\mathbb{Z}/(2^q))$. In (J), we are taking out $\xi(\hat{p}_1)$ as part of an ideal. We end up with

$$\begin{array}{lll} x^1\text{-torsion} & \mathbb{Z}/(2^{q-1})[\hat{p}_1]\{\alpha_i\hat{p}_1\} & 0 \le i < 4, \\ x^3\text{-torsion} & \mathbb{Z}/(2)[\hat{v}_1]\{\hat{v}_1 z, wz\} & \mathbb{Z}/(2)[\hat{p}_1]\{w\hat{p}_1, wz\hat{p}_1\}, \\ x^7\text{-torsion} & \mathbb{Z}/(2)[\hat{p}_1]\{\hat{p}_1, z\}. & \end{array}$$

To get this in the same form as $ER(2)^*(B\mathbb{Z}/(2^q))/(\mathrm{im}(N_*^{\mathrm{res}}))$, there is just one last step. The element $\hat{p}_1^2$ that seems to be x^7-torsion is the same, mod higher filtrations, as $\hat{v}_1\hat{p}_1$ (recall $\xi(\hat{p}_1)$ starts off as $\hat{v}_1\hat{p}_1 + \hat{v}_2\hat{p}_1^2$ mod (2)). This is x^3-torsion, so we should only have $\mathbb{Z}/(2)\{z, \hat{p}_1, z\hat{p}_1\}$ left as x^7-torsion, and we should have $\mathbb{Z}/(2)[\hat{p}_1]\{\hat{p}_1^2, z\hat{p}_1^2\}$ as x^3-torsion.

This shows we have an isomorphism of associated graded objects and completes the proof. ∎

A Appendix

Because we use this table all the time, it should be available for general reference. Here is $ER(2)^*$, written in its 48-periodic form, with $k \ge 0$ and $\alpha^0 = 1$.

15	$x\alpha^{3k+2}$	31	$x\alpha^{3k}$	47	$x\alpha^{3k+1}$
14	$x^2\alpha^{3k}$	30	$x^2\alpha^{3k+1}$	46	$x^2\alpha^{3k+2}$
13		29		45	x^3
12	$\alpha_3\alpha^{3k}$	28	$x^4,\ \alpha_3\alpha^{3k+1}$	44	$\alpha_3\alpha^{3k+2}$
11	x^5	27		43	
10		26		42	x^6
9		25		41	
8	$w\alpha^{3k+1}$	24	$\alpha_2,\ w\alpha^{3k+2}$	40	$w\alpha^{3k}$
	$2w\alpha^{3k+1} = \alpha_2\alpha^{3k+2}$		$2w\alpha^{3k+2} = \alpha_2\alpha^{3k+3}$		$2w\alpha^{3k} = \alpha_2\alpha^{3k+1}$
7	$xw\alpha^{3k+2}$	23	$xw\alpha^{3k}$	39	$xw\alpha^{3k+1}$
6	$x^2w\alpha^{3k}$	22	$x^2w\alpha^{3k+1}$	38	$x^2w\alpha^{3k+2}$
5		21		37	
4	$\alpha_1\alpha^{3k+1}$	20	$\alpha_1\alpha^{3k+2}$	36	$\alpha_1\alpha^{3k}$
3		19		35	
2		18		34	
1		17		33	
0	α^{3k}	16	α^{3k+1}	32	α^{3k+2}

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