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EPIMORPHISMS OF MODULES WHICH MUST BE ISOMORPHISMS

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Let R be an associative ring (not necessarily with identity).

DEFINITION 1. R is a left Π-ring if it has the following property: Let M be a finitely generated left R-module, N a submodule of M and $\phi:N\to M$ an epimorphism. Then ϕ is an isomorphism.

DEFINITION 2. R is a left Π_1-ring if it has identity and the following property: Let M be a finitely generated unitary left R-module, N a submodule of M and $\phi:N\to M$ an epimorphism. Then ϕ is an isomorphism.

If R is a ring let R_1 be the ring with identity obtained from R by adjoining the identity. We have $R_1\simeq R\oplus Z$ as abelian groups. If M is a left R-module then it can be also considered as a unitary left R_1-module and vice versa. Let E be a subset of M. The submodule of M generated by E is the intersection of all submodules of M which contain E. It follows that the submodule of M generated by E is the same for both module structures on M mentioned above. In particular, M is a finitely generated left R-module if and only if it is a finitely generated unitary left R_1-module.

It is clear that these remarks prove the first part of the following theorem.

THEOREM 1. *Let R be a ring. Then*

(i) *R is a left Π-ring if and only if R_1 is a left Π_1-ring.*

(ii) *If R has identity then it is a left Π-ring if and only if it is a left Π_1-ring.*

(iii) *Any homomorphic image of a left Π-ring (left Π_1-ring) is also a left Π-ring (left Π_1-ring).*

(iv) *A left Noetherian ring is a left Π-ring.*

Proof. (iii) follows from the fact that if S is a homomorphic image of R then every left S-module can be considered as a left R-module.

(iv) Let $\phi:N\to M$ be as in Definition 1. We want to prove that ϕ is an isomorphism. Let $\phi^0(0)=0$ and define by induction

$$\phi^{-n}(0) = \phi^{-1}(\phi^{-(n-1)}(0)), \qquad n = 1, 2, \ldots$$

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Then each $\phi^{-n}(0)$ is a submodule of N. We have in fact

$$\phi^{-n}(0) = \{x \mid x \in N, \phi(x) \in N, \ldots, \phi^{n-1}(x) \in N, \phi^n(x) = 0\}.$$

It follows that

$$0 = \phi^0(0) \subset \phi^{-1}(0) \subset \phi^{-2}(0) \subset \cdots$$

Since R is left Noetherian, and M is finitely generated if follows that M is Noetherian. There exists $k \geq 0$ such that $\phi^{-k}(0) = \phi^{-(k+1)}(0)$. We take k to be the smallest nonnegative integer with this property. Assume that $\phi^{-1}(0) \neq 0$ and so $k \geq 1$. Since $\phi^{-(k-1)}(0) \neq \phi^{-k}(0)$ there exists $x \in \phi^{-k}(0)$ such that $\phi^{k-1}(x) \neq 0$. But $x = \phi(y)$ for some $y \in \phi^{-(k+1)}(0)$ because ϕ is an epimorphism. Thus $\phi^{k-1}(x) = \phi^k(y) = 0$ because $y \in \phi^{-(k+1)}(0) = \phi^{-k}(0)$. This is a contradiction. Hence $\phi^{-1}(0) = 0$, i.e., ϕ is injective and consequently an isomorphism.

(ii) Assume that R is a left Π_1-ring and let $\phi: N \to M$ be as in Definition 1. We have $M = M_0 \oplus M_1$ where $RM_0 = 0$ and M_1 is a unitary left R-module. Also $N = N_0 \oplus N_1$ with $N_0 \subset M_0$ and $N_1 \subset M_1$. Since $\phi(N_0) \subset M_0$, $\phi(N_1) \subset M_1$ the restrictions $\phi_0: N_0 \to M_0$ and $\phi_1: N_1 \to M_1$ are epimorphisms. Since R is a left Π_1-ring ϕ_1 must be an isomorphism. Also ϕ_0 is an isomorphism because Z (the ring of integers) is Noetherian and we may use (iv). Hence ϕ is also an isomorphism.

In view of these results we can restrict to study only the left Π_1-rings. Our main result is the following:

THEOREM 2. *Any direct limit of left* Π_1*-rings is a left* Π_1*-ring.*

Proof. Let $A = \varinjlim A_i$ where A_i are left Π_1-rings. Let M be a finitely generated unitary left A-module

$$M = \sum_{k=1}^{n} Ax_i,$$

$N \subset M$ a submodule and $\phi: N \to M$ an epimorphism. Let $y_0 \in N$ be such that $\phi(y_0) = 0$. Choose $y_1, \ldots, y_n$ such that $\phi(y_k) = x_k$, $1 \leq k \leq n$. We may write

$$y_r = \sum_{k=1}^{n} a_{rk} x_k, \qquad 0 \leq r \leq n$$

with all a_{rk} in the image in A of a fixed A_{i_0}. Let M_{i_0} and N_{i_0} be the left A_{i_0}-modules obtained from M and N via the canonical homomorphisms $A_{i_0} \to A$. Let M_0 be the submodule of M_{i_0} generated by $x_1, \ldots, x_n$. It follows that $y_0, y_1, \ldots, y_n$ belong to M_0. Let N_0 be the submodule of N_{i_0} generated by $y_0, y_1, \ldots, y_n$. Then $N_0 \subset M_0$ and the restriction $\phi_0: N_0 \to M_0$ of ϕ is surjective. Since A_{i_0} is a left Π_1-ring ϕ_0 must be an isomorphism. Hence, $\phi_0(y_0) = \phi(y_0) = 0$ implies $y_0 = 0$.

COROLLARY 1. *Every commutative ring with identity is a left* Π_1*-ring.*

Proof. Such a ring is the direct limit of the direct system of its finitely generated subrings (containing the identity element). Every finitely generated subring of a commutative ring is Noetherian and so a left Π_1-ring.

This result was proved in [1]. Furthermore we have the following corollaries which correspond to Theorem 1, Corollaries 2–3 of [1].

COROLLARY 2. *Let R be a left Π_1-ring and M a unitary left R-module generated by n elements and N a free R-submodule of M of rank not less than n. Then M is a free R-module and rank N=rank M=n.*

The proof of this is straightforward.

COROLLARY 3. *Let R be a left Π_1-ring and $f:R\to S$ a homomorphism of rings. Assume that S has identity and that it is finitely generated as a left R-module via f. If $x, y \in S$ and $xy=1$ then $yx=1$.*

Proof. Let $\phi:S\to S$ and $\psi:S\to S$ be defined by $\phi(s)=sx$, $\psi(s)=sy$. Since $\psi \circ \phi=$ identity it follows that ψ is onto. Since R is a left Π_1-ring ψ must be an isomorphism and ϕ is its inverse. Hence $\phi \circ \psi=$identity which implies that $yx=1$.

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