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SHORT NOTES

GENERALIZATION OF HAEFELI'S CREEP-ANGLE ANALYSIS

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ABSTRACT. Using geometrical arguments, Haefeli developed a stress analysis for slabs of compressible viscous materials. His analysis was based on a key parameter called the creep angle. A generalization of the creep angle, called the deformation-rate coefficient, is derived by replacing geometrical arguments with continuum mechanics. Once the deformation-rate coefficient is found from *in situ* measurements, the stress field of the slab can be determined from a set of hyperbolic partial differential equations.

RÉSUMÉ. *Generalisation de l'analyse de l'angle de glissement de Haefeli.* A partir de considérations géométriques, Haefeli a réalisé une analyse des efforts mis en jeu pour des glissements en plaques de matériaux visqueux compressibles. Son analyse était fondée sur un paramètre clé appelé angle de glissement. Une généralisation de la notion d'angle de glissement, appelé coefficient de vitesse de déformation, a été élaborée en remplaçant les considérations géomètriques par des mécanismes continus. Une fois déterminé le coefficient de vitesse de déformation par des mesures *in situ*, le champ des efforts sur la plaque peut être calculé à partir d'une série d'équations différentielles partielles hyperboliques.

ZUSAMMENFASSUNG. *Verallgemeinerung der Kriechwinkel-Analyse von Haefeli.* Auf der Grundlage geometrischer Betrachtungen entwickelte Haefeli eine Belastungsanalyse für Scheiben von komprimierbarem, viskosem Material. Der Hauptparameter seiner Analyse ist der sogenannte Kriechwinkel. Eine Verallgemeinerung des Kriechwinkels, die "Koeffizient der Verformungsgeschwindigkeit" genannt wird, lässt sich herleiten, wenn man die geometrischen Betrachtungen durch Kontinuumsmechanik ersetzt. Ist der Koeffizient der Verformungsgeschwindigkeit durch Messungen *in situ* gefunden, so kann das Belastungsfeld der Scheibe aus einem Satz von hyperbolischen partiellen Differential-Gleichungen bestimmt werden.

I. INTRODUCTION

The creep-angle analysis was proposed by Haefeli (1963, 1967) as a practical method of obtaining the stress in the neutral zone of inclined planar slabs composed of compressible viscous materials such as snow or clay. By neutral zone, Haefeli meant that portion of the slab that is free from edge effects, essentially, the central region of the slab. With reference to Figure 1, the neutral zone is modeled as a planar slab extending to infinity in two directions, x and z. For this simple geometry, spatial gradients

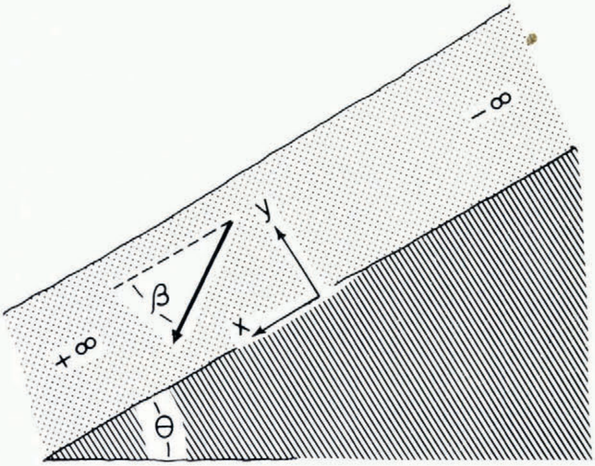

Fig. 1. Creep angle β in the neutral zone of a slab.

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exist only in the y-direction, that is, $\partial/\partial x$ and $\partial/\partial z$ are null. Furthermore, the neutral zone is assumed to be a region of plane deformation-rate, wherein the creep velocity in the z-direction is zero. The creep angle, β, in Haefeli's original treatment, is simply the angle formed by the x-axis and the trajectory of the moving snow particles.

Assuming coincidence of principal stress and principal deformation-rate axes, Haefeli utilized geometrical constructions to determine the neutral zone stress components as

$$t_{xy} = g \sin \theta \int_y^H \rho(y) \, \mathrm{d}y,$$

$$t_{yy} = -g \cos \theta \int_y^H \rho(y) \, \mathrm{d}y,$$

$$t_{xx} = t_{yy} + 2t_{xy} \tan \beta, \tag{1}$$

where ρ is the slab density, θ is the slab inclination to the horizontal, and H is the slab thickness. It will be the purpose of this short note to extend Haefeli's analysis to include non-neutral regions and non-planar slabs. A more detailed discussion of this particular problem, with application to snow slabs, is found in Perla (1971).

II. The deformation-rate coefficient

We choose a purely viscous constitutive law to relate the following slab variables: stress tensor t_{ij}, deformation-rate tensor d_{kl}, temperature T, and density ρ. The constitutive law is expressed in cartesian tensor notation as

$$t_{ij} = f_{ij}(d_{kl}, T, \rho) \tag{2}$$

where f_{ij} is a tensor function. A spatial reference system is attached to the slab's substratum at a convenient point, and d_{ij} is computed from the creep components u_i, according to

$$d_{ij} = \tfrac{1}{2}(u_{i,\,j} + u_{j,\,i}). \tag{3}$$

Material isotropy is assumed; Equation (2) must then reduce to (Truesdell and Noll, 1965)

$$t_{ij} = \phi_1 \delta_{ij} + \phi_2 d_{ij} + \phi_3 d_{ik} d_{kj} \tag{4}$$

where ϕ_i are functions of T, ρ, and the scalar invariants of d_{kl}. In Equation (4), the summation convention is observed for double subscripts, and δ_{ij} is the Kronecker delta. The expansion of Equation (4) for the components t_{xy}, t_{yy}, and t_{xx} is

$$\begin{aligned} t_{xy} &= \phi_2 d_{xy} + \phi_3 d_{xi} d_{iy}, \\ t_{yy} &= \phi_1 + \phi_2 d_{yy} + \phi_3 d_{yi} d_{iy}, \\ t_{xx} &= \phi_1 + \phi_2 d_{xx} + \phi_3 d_{xi} d_{ix}, \end{aligned} \tag{5}$$

where the summation is on i only. For plane deformation-rate problems, wherein d_{iz} vanishes, Equation (5) may be manipulated algebraically to give the simple relationship

$$t_{xx} = t_{yy} + 2Dt_{xy} \tag{6}$$

where $D(x, y)$, a quantity we call the deformation-rate coefficient, consists of

$$D(x, y) = \frac{d_{xx} - d_{yy}}{2d_{xy}} = \left(\frac{\partial u_x}{\partial x} - \frac{\partial u_y}{\partial y}\right) \Big/ \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right). \tag{7}$$

In the neutral zone, $\partial/\partial x$ is null, and hence Equation (7) reduces to

$$D(y) = -\frac{\mathrm{d}u_x}{\mathrm{d}u_y}. \tag{8}$$

In the special case, where u_x and u_y vary linearly with y, $D(y)$ represents the tangent of the creep angle.

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III. GENERALIZATION

Plane problems require simultaneous solution of Equation (6) and the equations of equilibrium which are

$$\left.\begin{aligned} \frac{\partial t_{xx}}{\partial x}+\frac{\partial t_{xy}}{\partial y}+\rho(y)\,g\sin\theta = 0,\\ \frac{\partial t_{xy}}{\partial x}+\frac{\partial t_{yy}}{\partial y}-\rho(y)\,g\cos\theta = 0. \end{aligned}\right\} \quad (9)$$

The set of Equations (6) and (9) are hyperbolic partial differential equations in the unknowns t_{xy}, t_{yy}, and t_{xx}. Solutions can be found for arbitrary slab geometries, provided $D(x, y)$ can be specified throughout the region of interest, and boundary conditions can be specified at the slab–atmosphere interface. Figure 2 illustrates the general problem. If $D(x, y)$ can be specified within the region bounded by NPMN, then it is possible to find the characteristics of Equations (6) and (9), namely NQ and MQ. The point Q must be contained within NPMN. If t_{xy} and t_{yy} can be specified along NM, then t_{xy}, t_{yy}, and t_{xx} can be determined uniquely within the region bounded by NQMN. This is the Cauchy problem in hyperbolic partial differential equations; numerical solutions are available (Panov, 1963).

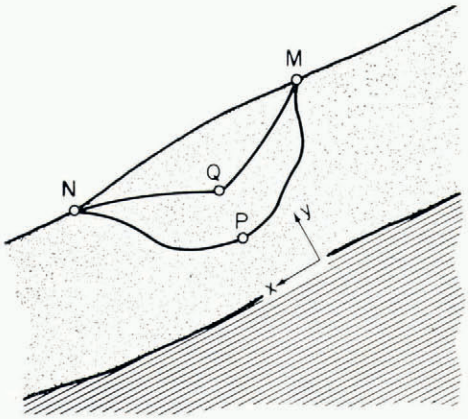

Fig. 2. Characteristics NQ *and* MQ *within a slab region of interest* NPMN.

The hyperbolic system, Equations (6) and (9), is linear and avoids the non-linear problems of the alternative, elliptical system which consists of Equations (4), (9), and the compatibility equations for plane deformation-rate. Moreover, solution of the elliptical system requires knowledge of the elusive phenomenological functions ϕ_i, and knowledge of the boundary conditions around the entire slab region of interest. In contrast, the hyperbolic system requires *in situ* measurement of $D(x, y)$. Such measurement seems feasible for a wide variety of natural slabs.

It is worthwhile to note that the above analysis could be repeated for deformations instead of deformation-rates. A deformation tensor could replace d_{kl} in Equation (2). The analysis would carry through with displacements replacing velocities, u_x and u_y. In this case, the experimental task is *in situ* measurements of deformations instead of deformation-rates.

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