Slide Presentation on the Riemann Hypothesis

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Abstract

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. In 2011, Patrick Solé and Michel Planat stated a new criterion for the Riemann Hypothesis. We prove the Riemann Hypothesis is true using this criterion.

Keywords

Riemann Hypothesis, Prime numbers, Chebyshev function, Riemann zeta function.

Objectives

The Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x, where log is the natural logarithm. We provide a proof for the Riemann Hypothesis using the properties of the Chebyshev function.

Say Dedekind(q_n) holds provided

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(q_n)$$

where the constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, q_n is the nth prime number and $\zeta(x)$ is the Riemann zeta function.

Theorem 1

Dedekind(q_n) *holds for all prime numbers* $q_n > 3$ *if and only if the Riemann Hypothesis is true (Solé and Planat, 2011).*

What if the Riemann Hypothesis were false?

Theorem 2

If the Riemann Hypothesis is false, then there are infinitely many prime numbers q_n for which $\mathrm{Dedekind}(q_n)$ *do not hold.*

We know the Riemann Hypothesis is false, if there exists some natural number $x_0 \geq 5$ such that $g(x_0) > 1$ or equivalent $\log g(x_0) > 0$ (Solé and Planat, 2011):

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \times \log \theta(x) \times \prod_{q \leq x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We know the bound (Solé and Planat, 2011):

$$\log g(x) \geq \log f(x) - \frac{2}{x}$$

where f is introduced in the Nicolas paper (Nicolas, 1983):

$$f(x) = e^{\gamma} \times \log \theta(x) \times \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

Remark

We know when the Riemann Hypothesis is false, then there exists a real number $b < \frac{1}{2}$ and there are infinitely many natural numbers x such that $\log f(x) = \Omega_{+}(x^{-b})$ (Nicolas, 1983).

According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}, y > y_0 \colon \log f(y) \geq k \times y^{-b}.$$

That inequality is equivalent to $\log f(y) \geq \left(k \times y^{-b} \times \sqrt{y}\right) \times \frac{1}{\sqrt{y}}$, but we know that

$$\lim_{y \to +\infty} \left(k \times y^{-b} \times \sqrt{y}\right) = +\infty$$

for every possible positive value of k when $b < \frac{1}{2}$.

In this way, this implies that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}, y > y_0 \colon \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Hence, if the Riemann Hypothesis is false, then there are infinitely many natural numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. Since $\frac{2}{x} = o(\frac{1}{\sqrt{x}})$, then it would be infinitely many natural numbers x_0 such that $\log g(x_0) > 0$ (Solé and Planat, 2011).

In addition, if $\log g(x_0) > 0$ for some natural number $x_0 \geq 5$, then $\log g(x_0) = \log g(q_n)$ where q_n is the greatest prime number such that $q_n \leq x_0$. In fact,

$$\prod_{q \leq x_0} \left(1 + \frac{1}{q}\right)^{-1} = \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right)^{-1}$$

and

$$\theta(x_0) = \theta(q_n)$$

according to the definition of the Chebyshev function. ■

We define $H = \gamma - B$ such that $B \approx 0.2614972128$ is the Meissel-Mertens constant (Choie et al., 2007). We know from the constant H, the following formula:

Theorem 3

We have that (Choie et al., 2007):

$$\sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \frac{1}{q_k} \right) = \gamma - B = H.$$

We know this value of the Riemann zeta function:

Theorem 4

It is known that (Edwards, 2001):

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6}.$$

A Key Theorem

Theorem 5

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k}) \right) = \log(\zeta(2)) - H.$$

We obtain that

$$\log(\zeta(2)) - H = \log(\prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1}) - H$$
$$= \sum_{k=1}^{\infty} \left(\log(\frac{q_k^2}{(q_k^2 - 1)}) \right) - H$$
$$= \sum_{k=1}^{\infty} \left(\log(\frac{q_k^2}{(q_k - 1) \times (q_k + 1)}) \right) - H$$
$$= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) + \log(\frac{q_k}{q_k + 1}) \right) - H$$

$$= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(\frac{q_k + 1}{q_k}) \right) - H$$

$$= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(1 + \frac{1}{q_k}) \right) - \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \frac{1}{q_k} \right)$$

$$= \sum_{k=1}^{\infty} \left(\log(\frac{q_k}{q_k - 1}) - \log(1 + \frac{1}{q_k}) - \log(\frac{q_k}{q_k - 1}) + \frac{1}{q_k} \right)$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k}) \right)$$

and the proof is done. ■

Theorem 6

Dedekind(q_n) *holds for all prime numbers* $q_n > 3$ *if and only if the inequality*

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x > q_n\}}(q_k)) \times \log(1 + \frac{1}{q_k}) \right) > B + \log \log \theta(q_n)$$

is satisfied for all prime numbers $q_n > 3$*, where the set* $S = \{x :\ x > q_n\}$ *contains all the real numbers greater than* q_n *and* χ_S *is the characteristic function of the set* S *(This is the function defined by* $\chi_S(x) = 1$ *when* $x \in S$ *and* $\chi_S(x) = 0$ *otherwise).*

When Dedekind(q_n) holds, we apply the logarithm to the both sides of the inequality:

$$\log(\zeta(2)) + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > \gamma + \log\log\theta(q_n)$$

$$\log(\zeta(2)) - H + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > B + \log\log\theta(q_n)$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - \log(1 + \frac{1}{q_k}) \right) + \sum_{q \leq q_n} \log(1 + \frac{1}{q}) > B + \log\log\theta(q_n).$$

Let's distribute the elements of the inequality to obtain that

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x > q_n\}}(q_k)) \times \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

when Dedekind(q_n) holds. The same happens in the reverse implication. ■

Theorem 7

The Riemann Hypothesis is true if the inequality

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n.

The inequality

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x > q_n\}}(q_k)) \times \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

is satisfied when

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x \geq q_n\}}(q_k)) \times \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

is also satisfied, where the set $S = \{x:\ x \geq q_n\}$ contains all the real numbers greater than or equal to q_n.

In the inequality

$$\sum_{k=1}^{\infty}\left(\frac{1}{q_k} - (\chi_{\{x:\ x \geq q_n\}}(q_k)) \times \log(1+\frac{1}{q_k})\right) > B + \log\log\theta(q_n)$$

only change the values of

$$\log(1+\frac{1}{q_n}) + \log\log\theta(q_n)$$

and

$$\log\log\theta(q_{n+1})$$

between the consecutive primes q_n and q_{n+1}. It is enough to show that

$$\log(1+\frac{1}{q_n}) + \log\log\theta(q_n) \geq \log\log\theta(q_{n+1})$$

for all sufficiently large prime numbers q_n.

Indeed, the inequality

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x \geq q_n\}}(q_k)) \times \log(1 + \frac{1}{q_k}) \right) > B + \log\log\theta(q_n)$$

is the same as

$$\sum_{k=1}^{\infty} \left(\frac{1}{q_k} - (\chi_{\{x:\ x \geq q_{n+1}\}}(q_k)) \times \log(1 + \frac{1}{q_k}) \right)$$

$$> B + \log\log\theta(q_{n+1}) + \log(1 + \frac{1}{q_n}) + \log\log\theta(q_n) - \log\log\theta(q_{n+1})$$

where q_n and q_{n+1} are consecutive primes.

If the Riemann Hypothesis is false, then

$$\log(1 + \frac{1}{q_n}) + \log\log\theta(q_n) \geq \log\log\theta(q_{n+1})$$

must be violated for infinitely many n's, since Dedekind(q_{n+1}) will not hold for infinitely many q_{n+1}'s. By contraposition, the Riemann Hypothesis should be true when the previous inequality is satisfied for all sufficiently large prime numbers q_n.

This is

$$\log\left((1+\frac{1}{q_n})\times\log\theta(q_n)\right)\geq\log\log\theta(q_{n+1}).$$

That is equivalent to

$$\log\log\theta(q_n)^{1+\frac{1}{q_n}}\geq\log\log\theta(q_{n+1}).$$

To sum up, the Riemann Hypothesis is true when

$$\theta(q_n)^{1+\frac{1}{q_n}}\geq\theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n. ■

Theorem 8

For all $n \geq 2$*, we have (Ghosh, 2019):*

$$\frac{\theta(q_n)}{\log q_{n+1}} \geq n \times (1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n}).$$

Theorem 9

For every $x \geq 19035709163$ *(Axler, 2018):*

$$\theta(x) > (1 - \frac{0.15}{\log^3 x}) \times x.$$

We define the prime counting function $\pi(x)$ as

$$\pi(x) = \sum_{p \leq x} 1.$$

We also know this property for the prime counting function:

Theorem 10

For every $x \geq 19027490297$ *(Axler, 2018):*

$$\pi(x) > \eta_x$$

where

$$\eta_x = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2 \times x}{\log^3 x} + \frac{5.85 \times x}{\log^4 x} + \frac{23.85 \times x}{\log^5 x} + \frac{119.25 \times x}{\log^6 x} + \frac{715.5 \times x}{\log^7 x} + \frac{5008.5 \times x}{\log^8 x}.$$

The Main Theorem

Theorem 11

The Riemann Hypothesis is true.

The Riemann Hypothesis is true when

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_{n+1})$$

is satisfied for all sufficiently large prime numbers q_n. That is the same as

$$\theta(q_n)^{1+\frac{1}{q_n}} \geq \theta(q_n) + \log(q_{n+1})$$

$$\theta(q_n)^{\frac{1}{q_n}} \geq 1 + \frac{\log(q_{n+1})}{\theta(q_n)}$$

after dividing both sides of the inequality by $\theta(q_n)$.

Using the known results, we only need to show that

$$\frac{\theta(q_n)}{\log q_{n+1}} \geq n \times (1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n})$$
$$> \eta_{q_n} \times (1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n})$$
$$> \frac{q_n}{\log q_n + \log(1 - \frac{0.15}{\log^3 q_n})}$$

for a sufficiently large prime number q_n where

$$\eta_{q_n} = \frac{q_n}{\log q_n} + \frac{q_n}{\log^2 q_n} + \frac{2 \times q_n}{\log^3 q_n} + \frac{5.85 \times q_n}{\log^4 q_n}$$
$$+ \frac{23.85 \times q_n}{\log^5 q_n} + \frac{119.25 \times q_n}{\log^6 q_n} + \frac{715.5 \times q_n}{\log^7 q_n} + \frac{5008.5 \times q_n}{\log^8 q_n}.$$

As q_n increases, $(1 - \frac{1}{\log n} + \frac{\log \log n}{4 \times \log^2 n})$ gets closer to 1 and η_{q_n} starts to become much greater than $\frac{q_n}{\log q_n + \log(1 - \frac{0.15}{\log^3 q_n})}$. However, this implies that

$$\frac{\log(1 - \frac{0.15}{\log^3 q_n}) + \log q_n}{q_n} > \frac{\log(q_{n+1})}{\theta(q_n)}$$

which is equal to

$$1 + \frac{\log(1 - \frac{0.15}{\log^3 q_n}) + \log q_n}{q_n} > 1 + \frac{\log(q_{n+1})}{\theta(q_n)}$$

for a sufficiently large prime number q_n.

It is also a known result that

$$\theta(q_n)^{\frac{1}{q_n}} > (1 - \frac{0.15}{\log^3 q_n})^{\frac{1}{q_n}} \times q_n^{\frac{1}{q_n}}$$

for a sufficiently large prime number q_n. In this way, we deduce that

$$\theta(q_n)^{\frac{1}{q_n}} \geq 1 + \frac{\log(q_{n+1})}{\theta(q_n)}$$

when the inequality

$$(1 - \frac{0.15}{\log^3 q_n})^{\frac{1}{q_n}} \times q_n^{\frac{1}{q_n}} \geq 1 + \frac{\log(1 - \frac{0.15}{\log^3 q_n}) + \log q_n}{q_n}$$

is satisfied for every sufficiently large prime number q_n.

We have that:

$$\frac{\log(1 - \frac{0.15}{\log^3 q_n}) + \log q_n}{q_n} \geq \log(1 + \frac{\log(1 - \frac{0.15}{\log^3 q_n}) + \log q_n}{q_n})$$

since

$$\frac{\log(1 - \frac{0.15}{\log^3 q_n}) + \log q_n}{q_n} > -1$$

for every sufficiently large prime number q_n. Certainly, if $x > -1$, then $x \geq \log(1 + x)$ (Kozma, 2022). We know that

$$\frac{\log(1 - \frac{0.15}{\log^3 q_n}) + \log q_n}{q_n} = \frac{\log\left((1 - \frac{0.15}{\log^3 q_n}) \times q_n\right)}{q_n}$$
$$= \log\left((1 - \frac{0.15}{\log^3 q_n})^{\frac{1}{q_n}} \times q_n^{\frac{1}{q_n}}\right)$$

by the properties of the logarithm.

This implies that

$$\log((1-\frac{0.15}{\log^3 q_n})^{\frac{1}{q_n}} \times q_n^{\frac{1}{q_n}}) \geq \log(1+\frac{\log(1-\frac{0.15}{\log^3 q_n})+\log q_n}{q_n})$$

which is equivalent to

$$(1-\frac{0.15}{\log^3 q_n})^{\frac{1}{q_n}} \times q_n^{\frac{1}{q_n}} \geq 1+\frac{\log(1-\frac{0.15}{\log^3 q_n})+\log q_n}{q_n}$$

for every sufficiently large prime number q_n. Putting all together yields the proof of the Riemann Hypothesis. ■

References

Christian Axler. New Estimates for Some Functions Defined Over Primes. *Integers*, 18, 2018. URL http://math.colgate.edu/~integers/s52/s52.pdf.

Aditya Ghosh. An asymptotic formula for the Chebyshev theta function. *Notes on Number Theory and Discrete Mathematics*, 25(4):1–7, 2019. URL https://nntdm.net/papers/nntdm-25/NNTDM-25-4-001-007.pdf.

László Kozma. Useful Inequalities. http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf, 2022. Accessed on June 2022.

Harold M. Edwards. *Riemann's Zeta Function*. Dover Publications, 2001. URL https://www.book-info.com/isbn/0-486-41740-9.htm.

YoungJu Choie, Nicolas Lichiardopol, Pieter Moree, and Patrick Solé. On Robin's criterion for the Riemann hypothesis. *Journal de Théorie des Nombres de Bordeaux*, 19 (2):357–372, 2007. URL https://jtnb.centre-mersenne.org/item/10.5802/jtnb.591.pdf.

Jean-Louis Nicolas. Petites valeurs de la fonction d'Euler. *Journal of number theory*, 17 (3):375–388, 1983. URL https://doi.org/10.1016/0022-314X(83)90055-0.

Patrick Solé and Michel Planat. Extreme values of the Dedekind ψ function. *Journal of Combinatorics and Number Theory*, 3(1):33–38, 2011. arXiv URL https://arxiv.org/pdf/1011.1825.pdf.