Basic concepts in viscous flow

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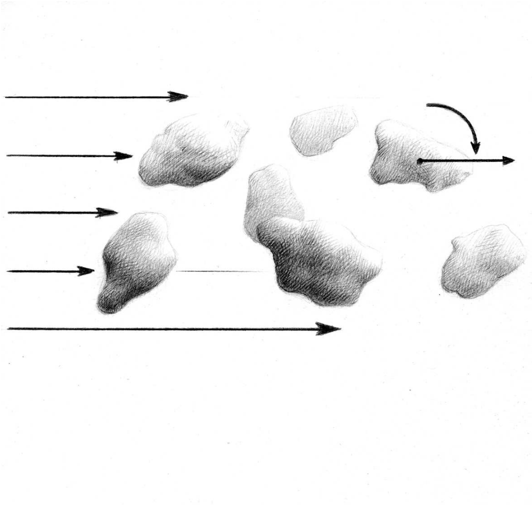
Microhydrodynamics

Microhydrodynamics, a term coined by G. K. Batchelor in the 1970s, deals with processes occurring in fluid flow when the characteristic length of the flow field is of the order of one micron

Many particles in a flow



Equations for an incompressible fluid

Continuity equation for an incompressible fluid

$$\nabla \cdot \mathbf{u} = 0$$

Equation for conservation of momentum

$$\begin{aligned} \rho[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}] &= \mathbf{f} + \nabla \cdot \boldsymbol{\sigma}^{\mathbf{a}} \\ &= \mathbf{f} - \nabla p^a + \mu \nabla^2 \mathbf{u} \end{aligned}$$

The superscript *a* indicates an absolute pressure and a corresponding absolute stress tensor. The term 'absolute stress' is used to indicate the actual stress (with the absolute pressure being the true pressure) rather than a modified stress to be defined in slide 12 in which the hydrostatic pressure is removed.

Newtonian fluid

Constitutive equation for a Newtonian fluid

$$\sigma_{ij}^a = -p^a \delta_{ij} + 2\mu e_{ij}$$

Symmetric stress tensor

$$\sigma_{ij}^a = \sigma_{ji}^a$$

Symmetric rate-of-strain tensor

$$e_{ij} = e_{ji} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Navier-Stokes equations

Incompressibility

$$\nabla \cdot \mathbf{u} = 0$$

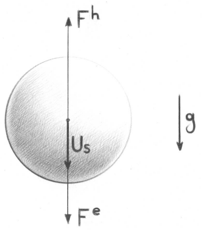
Momentum equation

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{f} - \nabla p^a + \mu \nabla^2 \mathbf{u}$$

Reynolds number

$$Re = \frac{|\rho(\mathbf{u}\cdot\nabla)\mathbf{u}|}{|\mu\nabla^2\mathbf{u}|} = \frac{UL}{\nu}$$

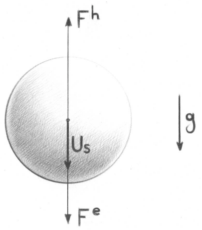
For sedimenting (spherical) grain of sand of size $L \sim 1\mu$m settling in water at velocity $U \sim 1\mu$m/s, $Re \sim 10^{-6}$



Stokes number

$$St = \frac{|\rho \partial \mathbf{u}/\partial t|}{|\mu \nabla^2 \mathbf{u}|} = \frac{L^2}{T\nu}$$

Sphere of radius $a = 1\mu$m sedimenting in water at stationary regime when time $T \gg a^2/\nu \sim 10^{-6}$ s



Stokes equations

Stokes equations: $Re \ll 1$ and $St \ll 1$

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \boldsymbol{\sigma}^a = -\nabla p^a + \mu \nabla^2 \mathbf{u} = \mathbf{f}$$

Homogeneous Stokes equations: $\mathbf{f} = \rho\,\mathbf{g} \Rightarrow p = p^a - \rho\,\mathbf{g} \cdot \mathbf{x}$

$$\nabla \cdot \mathbf{u} = 0$$

$$\nabla \cdot \boldsymbol{\sigma} = -\nabla p + \mu \nabla^2 \mathbf{u} = 0$$

Other expressions for Stokes equations

Stress tensor $\boldsymbol{\sigma}$

$$\sigma_{ij} = \sigma_{ji} = -p\delta_{ij} + 2\mu e_{ij}$$

Rate-of-strain tensor $\mathbf{e}$

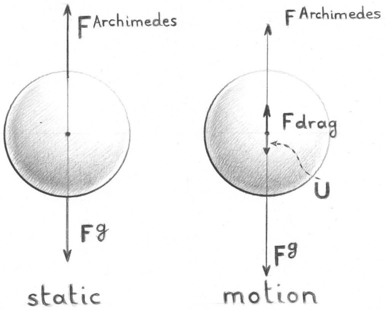
$$e_{ij} = e_{ji} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Stokes equations

$$\nabla \cdot \mathbf{u} = 0 \text{ or } e_{ii} = 0$$

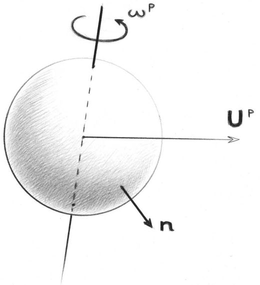
$$\nabla \cdot \boldsymbol{\sigma}^a = \mathbf{f} \text{ or } \frac{\partial \sigma^a_{ij}}{\partial x_j} = f_i$$

Buoyancy and drag



$$\mathbf{F}^{\mathbf{drag}} = \int_{S_p} (\boldsymbol{\sigma}^{\mathbf{a}} + \rho \mathbf{g} \cdot \mathbf{x} \mathbf{I}) \cdot \mathbf{n} \; dS = \int_{S_p} \boldsymbol{\sigma} \cdot \mathbf{n} \; dS = -4\pi a^3 (\rho_p - \rho) \mathbf{g}/3$$

Boundary conditions



No-slip boundary condition on the particles:

$$\mathbf{u}(\mathbf{x}) = \mathbf{U^p} + \boldsymbol{\omega}^\mathbf{p} \times (\mathbf{x} - \mathbf{x_p})$$

at the surface of a particle with center of mass at $\mathbf{x_p}$

+ Outer boundary condition on a containing vessel or at infinity

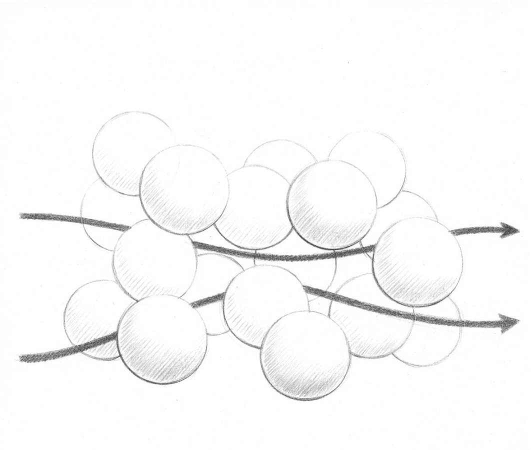
Linearity

Linearity of the Stokes equations: no non-linear convective acceleration term $(\mathbf{u} \cdot \nabla)\mathbf{u}$

- Principle of superposition: by adding different solutions of the Stokes equations one obtains also a solution of the Stokes equations
- Reversibility: the motion is reversible in the driving force

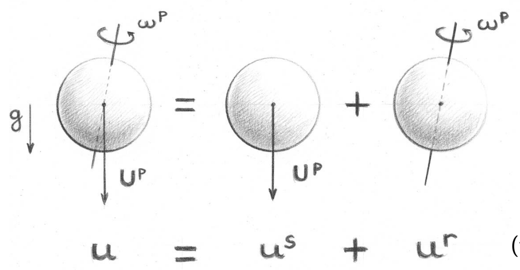
Streamlines in a porous medium

A doubling of the driving pressure gradient yields a doubling of the flow rate but no change to the streamlines



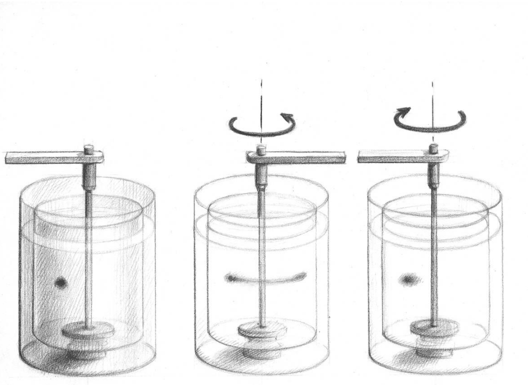
Principle of superposition

Summation of translation and rotation



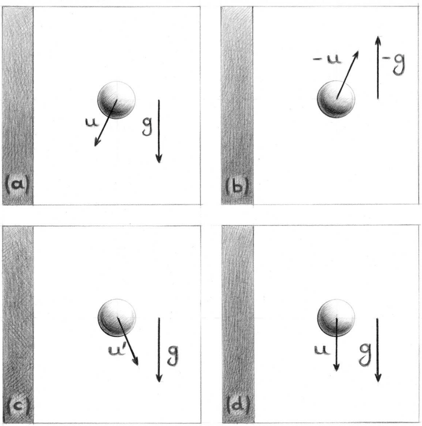
(fluid motions)

Description of G. I.Taylor film on reversibility



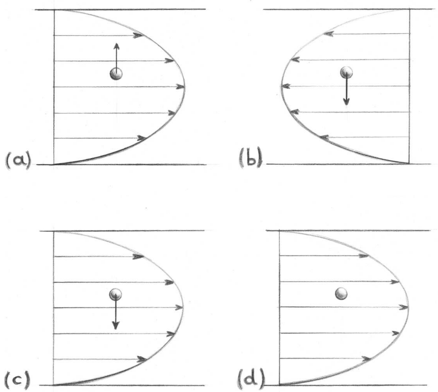
Reversibility argument for a sphere settling near a wall

A spherical particle falling adjacent to a wall falls at constant distance, as shown by the reversibility principle depicted visually here



Reversibility argument for a sphere in a Poiseuille flow

A single neutrally-buoyant spherical particle in Poiseuille flow stays at a fixed distance from the wall, as shown by the reversibility principle depicted visually here



Instantaneity

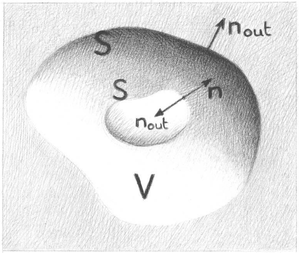
No history dependence $\partial \mathbf{u}/\partial t$

The flow is determined by the configuration given by the boundary conditions, coming both from the particle positions and outer boundaries

Information from boundary motion communicated to infinity instantly

- Divergence of the homogeneous momentum equation $\Rightarrow \nabla^2 p = 0$.
- Curl of the homogeneous momentum equation $\Rightarrow \nabla^2 \tilde{\boldsymbol{\omega}} = 0$ with $\tilde{\boldsymbol{\omega}} = \nabla \times \mathbf{u}$.
- Pressure p and vorticity $\tilde{\boldsymbol{\omega}}$ are harmonic

Kinetic energy balance



Kinetic energy:

$$K = \int_V \rho \frac{\mathbf{u}^2}{2} dV$$

where V is the fluid volume

Calculation for unsteady Stokes case:

$$\rho \frac{\partial \mathbf{u}}{\partial t} = \mathbf{f} + \nabla \cdot \boldsymbol{\sigma}$$

$$\begin{aligned} \frac{\partial K}{\partial t} &= \int_V \rho\, \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial t} dV = \int_V f_i u_i dV + \int_V u_i \frac{\partial \sigma_{ij}}{\partial x_j} dV \\ &= \int_V f_i u_i dV + \oint_S U_i \sigma_{ij} n_j^{out} dS - \int_V \frac{\partial u_i}{\partial x_j} \sigma_{ij} dV \end{aligned}$$

Rate of energy dissipation

Rate of energy dissipation due to viscosity

$$\Phi = \int_V \frac{\partial u_i}{\partial x_j}\sigma_{ij} dV = \int_V e_{ij}\sigma_{ij} dV = \int_V 2\mu e_{ij} e_{ij} dV \geq 0$$

Steady Stokes flow ($\partial K/\partial t = 0$)

$$\int_V \rho \mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} dV = \int_V f_i u_i dV + \oint_S U_i \sigma_{ij} n_j^{out} dS - \Phi = 0$$

Rate of energy dissipation = rate of working by external forces

$$\Phi = \int_V \mathbf{f} \cdot \mathbf{u}\, dV + \oint_S \mathbf{U} \cdot (\boldsymbol{\sigma} \cdot \mathbf{n}^{out}) dS$$

Minimum dissipation theorem

Consider $\mathbf{u}$ and $\mathbf{u}^S$ two velocity fields in the volume V such as

- $\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{u}^S = 0$ in the volume V
- $\mathbf{u} = \mathbf{u}^S = \mathbf{U}$ on the surface S limiting V
- $\mathbf{u}^S$ satisfying the homogeneous Stokes equations (no external force $\mathbf{f}$)

The minimum dissipation theorem states that the Stokes flow corresponds to the least dissipation

$$2\mu \int_V e_{ij}^S e_{ij}^S dV \leq 2\mu \int_V e_{ij} e_{ij} dV$$

Proof of minimum dissipation theorem

- Writing $\delta \mathbf{u} = \mathbf{u} - \mathbf{u}^S$ and $\delta e_{ij} = e_{ij} - e_{ij}^S$ with boundary conditions $\delta \mathbf{u} = 0$
- Difference between the two integrals

$$
\begin{aligned}
2\mu \int_V (e_{ij}e_{ij} - e_{ij}^S e_{ij}^S) dV &= 2\mu \int_V \delta e_{ij}(e_{ij} + e_{ij}^S) dV \\
&= \underbrace{2\mu \int_V \delta e_{ij}\delta e_{ij} dV}_{\geq 0} + \underbrace{4\mu \int_V \delta e_{ij} e_{ij}^S dV}_{=0}
\end{aligned}
$$

- Second integral

$$
4\mu \int_V \delta e_{ij} e_{ij}^S dV = 2 \int_V \delta e_{ij} \sigma_{ij}^S dV \quad \text{using } \delta e_{kk} = 0
$$

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- Second integral

$$4\mu \int_V \delta e_{ij} e_{ij}^S dV = 2 \int_V \frac{\partial \delta u_i}{\partial x_j} \sigma_{ij}^S dV \quad \text{using } \sigma_{ij}^S = \sigma_{ji}^S$$

Proof of minimum dissipation theorem

- Writing $\delta \mathbf{u} = \mathbf{u} - \mathbf{u}^S$ and $\delta e_{ij} = e_{ij} - e^S_{ij}$ with boundary conditions $\delta \mathbf{u} = 0$
- Difference between the two integrals

$$
\begin{aligned}
2\mu \int_V (e_{ij}e_{ij} - e^S_{ij}e^S_{ij})dV &= 2\mu \int_V \delta e_{ij}(e_{ij} + e^S_{ij})dV \\
&= \underbrace{2\mu \int_V \delta e_{ij}\delta e_{ij} dV}_{\geq 0} + \underbrace{4\mu \int_V \delta e_{ij} e^S_{ij} dV}_{=0}
\end{aligned}
$$

- Second integral

$$
4\mu \int_V \delta e_{ij} e^S_{ij} dV = -2 \int_V \delta u_i \frac{\partial \sigma^S_{ij}}{\partial x_j} dV + 2 \int_V \frac{\partial \delta u_i \sigma^S_{ij}}{\partial x_j} dV
$$

Proof of minimum dissipation theorem

- Writing $\delta\mathbf{u} = \mathbf{u} - \mathbf{u}^S$ and $\delta e_{ij} = e_{ij} - e_{ij}^S$ with boundary conditions $\delta\mathbf{u} = 0$
- Difference between the two integrals

$$\begin{aligned} 2\mu\int_V (e_{ij}e_{ij} - e_{ij}^S e_{ij}^S)dV &= 2\mu\int_V \delta e_{ij}(e_{ij} + e_{ij}^S)dV \\ &= \underbrace{2\mu\int_V \delta e_{ij}\delta e_{ij}dV}_{\geq 0} + \underbrace{4\mu\int_V \delta e_{ij}e_{ij}^S dV}_{=0} \end{aligned}$$

- Second integral

$$4\mu\int_V \delta e_{ij}e_{ij}^S dV = 2\int_S \delta u_i \sigma_{ij}^S n_j^{out} dS \quad \text{using } \frac{\partial \sigma_{ij}^S}{\partial x_j} = 0$$

Proof of minimum dissipation theorem

- Writing $\delta\mathbf{u} = \mathbf{u} - \mathbf{u}^S$ and $\delta e_{ij} = e_{ij} - e_{ij}^S$ with boundary conditions $\delta\mathbf{u} = 0$
- Difference between the two integrals

$$
\begin{aligned}
2\mu \int_V (e_{ij}e_{ij} - e_{ij}^S e_{ij}^S) dV &= 2\mu \int_V \delta e_{ij}(e_{ij} + e_{ij}^S) dV \\
&= \underbrace{2\mu \int_V \delta e_{ij}\delta e_{ij} dV}_{\geq 0} + \underbrace{4\mu \int_V \delta e_{ij} e_{ij}^S dV}_{=0}
\end{aligned}
$$

- Second integral

$$
4\mu \int_V \delta e_{ij} e_{ij}^S dV = 0 \quad \text{using boundary conditions } \delta\mathbf{u} = 0
$$

Uniqueness of the Stokes equation

- $(\mathbf{u}^{(1)}, p^{(1)})$ and $(\mathbf{u}^{(2)}, p^{(2)})$ two solutions of the homogeneous Stokes equation satisfying the same boundary conditions $\mathbf{u}^{(1)} = \mathbf{u}^{(2)} = \mathbf{U}$ on the surface S limiting the volume V
- $(\mathbf{u}^{(1)}, p^{(1)})$ Stokes flow $(\Rightarrow \Phi^{(1)} \leq \Phi^{(2)})$ and $(\mathbf{u}^{(2)}, p^{(2)})$ Stokes flow $(\Rightarrow \Phi^{(2)} \leq \Phi^{(1)}) \Rightarrow \Phi^{(1)} = \Phi^{(2)}$
- $\Phi^{(1)} = \Phi^{(2)} \Rightarrow e_{ij}^{(1)} = e_{ij}^{(2)}$
- $e_{ij}^{(1)} = e_{ij}^{(2)} \Rightarrow \mathbf{u}^{(1)} - \mathbf{u}^{(2)}$ solid body motion
- Boundary conditions $\mathbf{u}^{(1)} - \mathbf{u}^{(2)} = 0 \Rightarrow \mathbf{u}^{(1)} = \mathbf{u}^{(2)}$
- Homogeneous Stokes equations $\Rightarrow p^{(1)} - p^{(2)} =$ constant

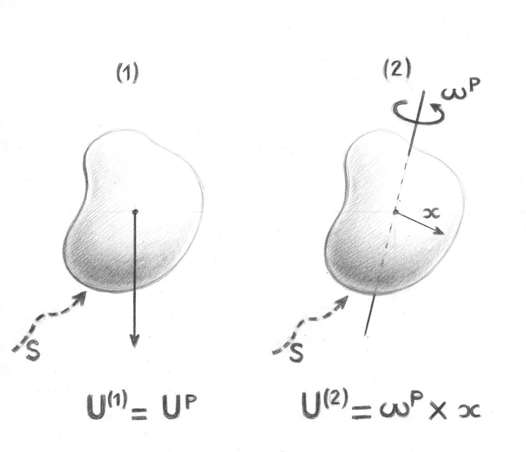
Reciprocal theorem

- $(\mathbf{u}^{(1)}, \boldsymbol{\sigma}^{(1)})$ and $(\mathbf{u}^{(2)}, \boldsymbol{\sigma}^{(2)})$ Stokes flows driven respectively by the external forces $\mathbf{f}^{(1)}$ and $\mathbf{f}^{(2)}$ and by the boundary conditions $\mathbf{u}^{(1)} = \mathbf{U}^{(1)}$ and $\mathbf{u}^{(2)} = \mathbf{U}^{(2)}$ on the surface S limiting the fluid volume V
- The reciprocal theorem states that the rate of working by the flow $\mathbf{u}^{(2)}$ against the forces of flow $\mathbf{u}^{(1)}$ ($\mathbf{f}^{(1)}$ in the volume and $\boldsymbol{\sigma}^{(1)} \cdot \mathbf{n}^{out}$ on the surface) is the same by interchanging (1) and (2)

$$\int_V f_j^{(1)} u_j^{(2)} dV + \oint_S \sigma_{ij}^{(1)} U_j^{(2)} n_i^{out} dS = \int_V f_j^{(2)} u_j^{(1)} dV + \oint_S \sigma_{ij}^{(2)} U_j^{(1)} n_i^{out} dS$$

Example of flows for a general shape body

Translation without rotation (left) and rotation without translation (right)



Proof of reciprocal theorem

Writing the left-hand side of equation with use of divergence theorem as

$$\begin{aligned}
\int_V f_j^{(1)} u_j^{(2)} dV + \oint_S \sigma_{ij}^{(1)} U_j^{(2)} n_i^{out} dS &= \int_V (f_j^{(1)} u_j^{(2)} + \frac{\partial(\sigma_{ij}^{(1)} u_j^{(2)})}{\partial x_i}) dV \\
&= \int_V (f_j^{(1)} u_j^{(2)} + \frac{\partial \sigma_{ij}^{(1)}}{\partial x_i} u_j^{(2)} + \sigma_{ij}^{(1)} \frac{\partial u_j^{(2)}}{\partial x_i}) dV \\
&= \int_V \sigma_{ij}^{(1)} \frac{\partial u_j^{(2)}}{\partial x_i} dV \quad \text{using } \frac{\partial \sigma_{ij}^{(1)}}{\partial x_i} + f_j = 0 \\
&= \int_V \sigma_{ij}^{(1)} e_{ij}^{(2)} dV \quad \text{using } \sigma_{ij}^{(1)} = \sigma_{ji}^{(1)} \\
&= 2\mu \int_V e_{ij}^{(1)} e_{ij}^{(2)} dV \quad \text{using } e_{kk}^{(1)} = 0
\end{aligned}$$

which, being symmetric, is also equal to the right-hand side of equation

Reciprocal theorem without external forces

Without external forces, the reciprocal theorem becomes

$$\oint_S \sigma_{ij}^{(1)} U_j^{(2)} n_i^{out} dS = \oint_S \sigma_{ij}^{(2)} U_j^{(1)} n_i^{out} dS$$

Movie References

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