2
Examples of Aspherical Manifolds

This chapter discusses some of the basic examples of, mainly closed, aspherical manifolds that give content to our inquiry. After all, what good would the Borel conjecture be if there were no aspherical manifolds?

We give some constructions of ones that come from locally symmetric manifolds (i.e., Lie theory) including both arithmetic and non-arithmetic examples, and also of others that do not.

By contrast, the construction of noncompact aspherical manifolds is quite easy. There is an open aspherical manifold with fundamental group π iff π is countable and has finite cohomological dimension, as one can see by thickening a finite-dimensional $K(\pi, 1)$ complex (i.e. replacing all the cells in a CW-decomposition by handles). Remarkably, aside from finiteness conditions, this characterizes the groups that are retracts of (fundamental groups of) aspherical manifolds.

2.1 Low-Dimensional Examples

In low dimensions, almost all connected manifolds (even noncompact) are aspherical. The only connected nonaspherical surfaces are the sphere and the projective plane.

In dimension 3, among closed orientable 3-manifolds *all are aspherical* unless one of the following very good reasons holds:

(1) the fundamental group is finite (in which case, the universal cover is $\mathcal{S}^3$ and the deck group is a subgroup of SO(3));
(2) the manifold is a nontrivial connected sum (and the separating 2-sphere is a nontrivial element of π_2); or
(3) the manifold is $\mathcal{S}^1 \times \mathcal{S}^2$.

https://doi.org/10.1017/9781316529645.003 Published online by Cambridge University Press

All of this[1] is a consequence of the sphere theorem of Papakyriakopoulos (see e.g. Hempel 1976; Jaco 1980)

However, in understanding even closed 3-manifolds, it is essential that one consider manifolds with nonempty boundary as part of the story. Given an arbitrary 3-manifold, one first has a decomposition into irreducible pieces, under connected sum. This is unique up to the order of the decomposition. Then one breaks the manifold summands further into pieces, where the gluing is done along certain embedded incompressible[2] tori. This topological decomposition was discovered by Jaco and Shalen, and Johannson, and explained geometrically by Thurston and Perelman: After breaking the 3-manifold along this decomposition along a set of canonical tori (its torus decomposition), one is left with pieces, *all of which have geometric structure*,[3] i.e. a manifold with a complete metric, which is locally homogeneous.

Let's be more concrete. Suppose we start with a knot K in $\mathcal{S}^3$, i.e. a smooth submanifold diffeomorphic to $\mathcal{S}^1$. The complement is always aspherical (as before, by Papakyriakopoulos's sphere theorem). For the unknot, the complement is $\mathcal{S}^1 \times \mathbb{R}^2$. It is often convenient to remove tubular neighborhoods of submanifolds, to obtain the "closed complement";[4] then we would obtain $\mathcal{S}^1 \times \mathcal{D}^2$.

For all knots, we obtain an aspherical manifold with boundary as its complement X, whose boundary is a torus. The unknot is characterized by the property that $\pi_1(\partial X) \to \pi_1(X)$ is not injective: a nontrivial knot always has an incompressible torus embedded in its complement (i.e. an embedded $\mathbb{T}^2$ so that π_1 injects).

Sometimes there is another torus (i.e. not isotopic to the boundary) in the complement that is incompressible. When this happens, essentially what that means is that this knot can be thought of as being wrapped around another knot, i.e. that it has a companion (Figure 2.1). The process of finding companions must end – although not obvious, there is a geometric complexity that increases under companionship.

1 At least for an infinite fundamental group. The description of what happens for a finite fundamental group depends on Perelman's solution of the geometricization conjecture.

2 Recall that a surface in a 3-manifold is incompressible if its normal bundle is trivial, and its fundamental group injects into the fundamental group of the manifold.

3 This is the celebrated geometricization conjecture. Actually, if an irreducible connected manifold contains any incompressible surfaces (and, in particular, if it has a nontrivial torus decomposition), then the geometricization of all of the pieces in its decomposition is a theorem of Thurston. For references, see the notes in §2.4.

4 There are subtleties with doing this in the topological category. In the setting of locally flat manifolds, everything works the same (see Kirby and Siebenmann, 1977). When we discuss orbifolds, we will see that the analogous issue is not solvable in the topological setting, i.e. one cannot always find a "closed regular neighborhood" of the knot, and one needs a substitute for tubular neighborhood theory.

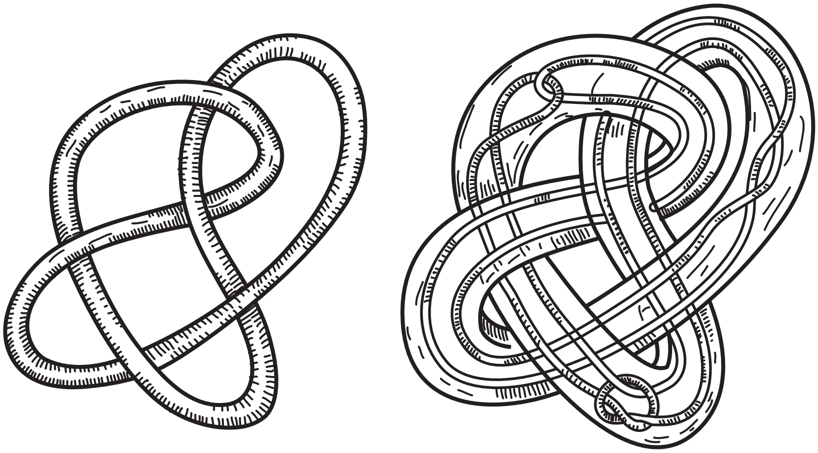

Figure 2.1 The knot on the left is a companion of the thinner knot on the right.[6]

So, now consider one of the deepest pieces, i.e. a knot with no companions. In that case, there are two cases: Torus knots are knots that lie on the surface of a torus that surrounds the unknot. These are parameterized by pairs of coprime integers (p, q) representing the homology class of the associated circles. *All of the remaining knots have hyperbolic complements*, i.e. have complete metrics of constant negative curvature and finite volume. (One can distinguish the two cases easily: the torus knots have fundamental group of their complement with nontrivial center – which precludes having a metric of negative curvature.[7])

In other words, the fundamental group of the complement Γ is naturally a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$.

The same is true for the annular regions between the various embedded tori: they all have hyperbolic structures. Thus, a typical knot complement (and according to geometricization, this is typical) is a union of hyperbolic (or perhaps one of several other geometries (see Scott, 1983) manifolds glued together along their cusps. (See Chapter 3 for more of a discussion of the geometry at ∞ of noncompact locally symmetric spaces.)

This union itself does not have a locally homogeneous structure. Its fundamental group cannot be a lattice in any Lie group.

This is because, in any of the three-dimensional geometries (see Scott, 1983),

6 Adapted from Thurston (1982).

7 This is Preissman's theorem, which can be found in most introductory differential geometry textbooks. See Bridson and Haefliger (1999) for a proof not using *differential* geometry.

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any $\mathbb{Z}^2$ is either *peripheral*, i.e. conjugate to the fundamental group of a boundary component,[8] or contains an element of the center of the fundamental group.[9] The $\mathbb{Z}^2$ coming from the torus of "companionship" is neither, and therefore this manifold does *not* have a locally symmetric structure.

The upshot is that it is very easy to obtain closed aspherical 3-manifolds whose fundamental groups are not lattices, e.g. the double of any knot complement other than torus knots. But they are obtained indirectly by gluing together lattices.

It is hard to make this precise, but till the early 1980s there was a general feeling that perhaps, somehow, lattices were the source of all closed aspherical manifolds. We will see that this is not the case as we go along, but let us start with the lattices themselves. Before we do, let us close this discussion by making one very useful observation about gluing aspherical objects:

Proposition 2.1 *Suppose that A, X, and Y are aspherical, $A = X \cap Y$, and that $\pi_1(A) \to \pi_1(X)$ and $\pi_1(A) \to \pi_1(Y)$ are injective. Then $X \cup Y$ is aspherical.*

Without the injectivity, the 2-sphere is a counterexample: it is a union of two disks along a circle, all aspherical, but not π_1 injective.

To see why the proposition is true, we shall construct the universal cover of $X \cup Y$ and *observe* that it is contractible. We begin by taking the cover of X. Over A (by injectivity) we get many copies of the universal cover of A (according to the cosets of $\pi_1(A) \to \pi_1(X)$). Each of these is glued to a copy of the universal cover of Y (which also contains many copies of the universal cover of A). We then proceed by gluing back copies of the universal cover of X, and so on. This is a union of contractible spaces glued together along (disjoint) contractible spaces, so this is contractible.

Remark 2.2 If one shrinks each copy of the universal cover of X to a point, and each copy of the universal cover of Y to a point while stretching and shrinking the copies of the universal cover of A to intervals, we get the Bass–Serre tree associated to this amalgamated free product description of $\pi_1 X \cup Y$.

This proposition is of critical importance. It enables us to construct interesting examples by gluing. We will either explicitly or tacitly apply it many times. A consequence of this is that we can take geometric models for given groups (i.e. $K(\pi, 1)$s for groups) and glue them together to construct models for various amalgamated free products and Higman–Neumann–Neumann (HNN)

8 Quotients by lattices have a natural compactification (the Borel–Serre compactification) which makes them into the interiors of manifolds with boundary. It is this virtual boundary that I am referring to when I describe a subgroup of the fundamental group as being peripheral.

9 Like in the situation of a circle bundle over a surface.

extensions: the quality of the union will depend on the quality of the complexes we begin with and of the inclusion of the subgroup. But, for example, it shows that the category of finite $K(\pi, 1)$, i.e. πs that are realized by finite aspherical complexes, is closed under amalgamated free products and HNN extensions.

This, in particular, allows the construction of finite aspherical complexes whose fundamental groups have unsolvable word problems, or other logical complications. The Davis construction, discussed below, will incorporate these features into fundamental groups of aspherical manifolds.

2.2 Constructions of Lattices

. . . Arithmetic and Non-arithmetic

Given a Lie group, even a quite explicit one like $\mathrm{O}(n, 1)$ (the automorphisms of the quadratic form ${x_1}^2 + {x_2}^2 + \cdots + {x_n}^2 - {x_{n+1}}^2$, i.e. the isometry group of hyperbolic n-space) or $\mathrm{SL}_n(\mathbb{R})$, it is not trivial to find uniform lattices; that is discrete subgroups of G such that G/Γ is compact.[10] Indeed, this is not always possible, e.g. for solvable Lie groups.[11]

However, if G is semisimple, Borel gave a general construction of uniform lattices (and Raghanuthan gave non-uniform lattices;[12] see Raghunathan (1972) for both). For $\mathrm{SL}_n(\mathbb{R})$ there is an obvious lattice, namely $\mathrm{SL}_n(\mathbb{Z})$, but it is not uniform, i.e. cocompact. If we think of $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ as the space of flat tori (as in §1.1), then tori that are more and more eccentric (i.e. the result of identifying opposite sides in a rectangle with sides t and $1/t$) leave any compact subset of this space (the shortest geodesic is approaching 0 length).

Let's make this a bit more precise (or more general). To talk about the "integer points" in a Lie group, we should define it over the field $\mathbb{Q}$ (there are many distinct ways of doing this). Then, for simplicity, let's assume that the group is linear – there will be a maximal subgroup isomorphic to $\mathbb{Q}^{*k}$ in $G(\mathbb{Q})$; here the diagonal matrices and $k = n - 1$. If $k > 0$, then $G/G(\mathbb{Z})$ is not compact and one can take powers of a matrix in this $\mathbb{Q}$-split torus to leave any compact.

The converse holds, i.e. the nonexistence of such a $\mathbb{Q}$-split torus implies compactness (and this is a theorem of Borel and Harish-Chandra). We defer

[10] Recall that Γ is a lattice in G if, giving G its natural (Haar) measure, the quotient G/Γ has finite volume.

[11] The two-dimensional Lie group of affine isomorphisms of $\mathbb{R} \to \mathbb{R}$ (the "$ax + b$" group) contains no lattices.

[12] Note that $\mathbb{R}^n$ has uniform lattices, but no non-uniform lattices. The same is true for all nilpotent real Lie groups.

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further discussion of this to Chapter 3, where the size of the torus will be seen to govern the "size" of G/Γ.

Another way to tell if a lattice is non-uniform is to see if it contains any nontrivial unipotent elements. (Consider the Lie group G as a matrix group, and then g is unipotent if its characteristic polynomial is $(t-1)^n$ for some n, i.e. if g differs from the identity by a nilpotent matrix.) No uniform lattice contains unipotent elements: the length of a geodesic represented by g in $\Gamma\backslash G/K$ is proportional to the supremum of $|\log(\lambda)|$ over eigenvalues of g representing the g. The converse had been a conjecture of Selberg, proved by Kazhdan and Margulis (and we refer to Margulis (1991) for the proof), and this property is often easy to check.

Finding appropriate $\mathbb{Q}$ structures for the case of SL_n is rather nontrivial and requires some development of the theory of division algebras. We shall leave this to the references, but for those who know some algebra, the group of units in an order in a division algebra of dimension n^2 does the trick.

Let us now return to the problem of constructing uniform lattices.

For $\mathrm{O}(n,1)$, looking at $\mathrm{O}(n,1)(\mathbb{Z})$ does not do the trick: one obtains a lattice, but not a uniform one.[13] However if we replace the quadratic form ${x_1}^2 + {x_2}^2 + \cdots + {x_n}^2 - {x_{n+1}}^2$ by $\mathbf{Q} = {x_1}^2 + {x_2}^2 + \cdots + {x_n}^2 - \sqrt{p}\, {x_{n+1}}^2$, then the real Lie group is the same: the quadratic forms are isomorphic over $\mathbb{R}$. However, $\mathrm{O}(\mathbb{Q})(\mathbb{Q}[\sqrt{p}])$ has two different embeddings into real orthogonal groups, associated to the two embeddings of $\mathbb{Q}[\sqrt{p}]$ into $\mathbb{R}$, according to whether $\sqrt{p}$ is positive or negative.

The (real) orthogonal group associated to making $\sqrt{p}$ negative is the usual compact orthogonal group. Note that the orthogonal group has no nontrivial unipotent elements. This means that $\mathrm{O}(\mathbb{Q})(\mathbb{Z}[\sqrt{p}])$ is a uniform lattice in $\mathrm{O}(n,1) \times \mathrm{O}(n+1)$. However, we can safely project to the first factor, as the second factor is compact, with at most a finite subgroup as kernel. In other words, the space $\mathrm{O}(\mathbb{Q})(\mathbb{Z}[\sqrt{p}])\backslash\mathrm{O}(n,1)/\mathrm{O}(n+1)$ is a compact hyperbolic orbifold. Replacing $\mathrm{O}(\mathbb{Q})(\mathbb{Z}[\sqrt{p}])$ by a torsion free subgroup of finite index gives a compact hyperbolic manifold.

This method produces many lattices. Lattices produced in this way are called *arithmetic*. Note that when written in coordinates, automorphisms defined using larger fields than $\mathbb{Q}$ give rise to Lie groups over $\mathbb{Q}$ – this is formally called "restriction of scalars." Using suitable quadratic forms over arbitrary totally real fields, we can get uniform lattices in any $\mathrm{O}(p,q)$.

The general case follows, as Borel says, from the statement that "any real

[13] Considering the automorphisms of the slight variant $a_1{x_1}^2 + \cdots + a_n{x_n}^2 - a_{n+1}{x_{n+1}}^2$, one obtains a uniform lattice iff this indefinite quadratic form does not represent 0 (i.e. does not vanish on any integral vector). However, the Hasse–Minkowski theorem says that this does not happen when $n > 4$.

semisimple Lie algebra has a form defined over a totally real field $E \neq \mathbb{Q}$ all of whose conjugates are compact." Borel proves this Lie-theoretic statement via tricky (for me) Lie algebra calculations in his paper (and the book of Raghunathan (1972) explains how to guarantee $\mathbb{Q}$ forms that produce the non-uniform lattices, as well).

For simple Lie groups of rank ≥ 2 (or even irreducible[14] lattices in semisimple groups) Margulis shows that these are all the examples, i.e. that all lattices are arithmetic. The reader should pause to reflect on how amazing this result is: one is given a structure with only local information defined over $\mathbb{R}$ (say a group of real matrices, or a finite-volume Riemannian manifold modeled on some $K\backslash G$) and one needs to find an algebraic number field and a form of the Lie group from this and then an isomorphism of one's given object with the arithmetic construction.

In the cases not excluded by Margulis (and the subsequent work of Corlette, Gromov, and Schoen that proves arithmeticity in some rank-1 situations by more analytic methods: see Gromov and Schoen, 1992), it is an important question of whether there are non-arithmetic lattices.

We mention here three such constructions, all of which are in $\mathrm{O}(n, 1)$. (Some examples are also known in $\mathrm{U}(n, 1)$ for small values of n (see, e.g., Deligne and Mostow, 1993), but these are isolated.)

2.2.1 Method One: Reflection Groups

The first is classical, and is based on constructing polyhedra in hyperbolic space so that reflections across its walls generate a reflection group on hyperbolic space. In the hyperbolic plane, the easiest example is a triangle with angles π/p, π/q, and π/r, so that $1/q + 1/r < 1$. (Below is an example with $p, q, r = 2, 3, 9$.) Even in dimension 2, Takeuchi (1977) showed only finitely many of these are arithmetic (and indeed gave a list of them).

It is known that such examples exist in small dimension, and do not exist in very high dimensions. Nevertheless, they perhaps motivate the Davis construction to be discussed in §2.3 below. Figure 2.2 shows a nice hyperbolic planar group generated by reflections.

[14] A lattice is reducible if, after passing to a subgroup of finite index, it is a product of two other lattices. An irreducible lattice in a product of real groups will project to a dense subgroup of each of the factors. So, for example, among lattices acting on a product of two hyperbolic planes, reducible ones will have deformations, but irreducible ones will be arithmetic (and have no deformations).

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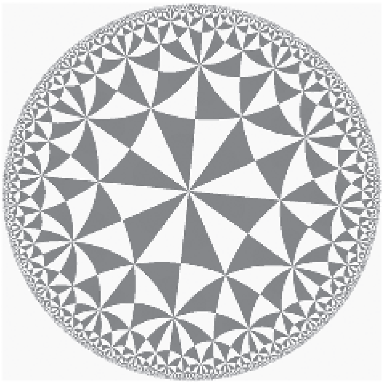

Figure 2.2 A hyperbolic triangle group.

2.2.2 Method Two: Closing Cusps

This method is due to Thurston, and is his famous Dehn surgery theorem (see Thurston, 2002). Consider a hyperbolic manifold with cusps (e.g. a knot complement, or most[15] link complements for "nonsplittable" links, i.e. links in which components cannot be isotoped to lie in disjoint balls). Thurston shows that for all "sufficiently large" surgeries, one obtains a compact hyperbolic manifold.

What does this mean? Given a manifold whose boundary is a torus, we can "close it up" by gluing in a solid torus $\mathcal{S}^1 \times \mathcal{D}^2$. Although there is an $\mathrm{SL}_2(\mathbb{Z}) = \pi_0 \mathrm{Diff}(\mathbb{T}^2)$ set of possible gluing diffeomorphisms, the diffeomorphism type of the manifold is determined by the image of the circle $\partial\mathcal{D}^2$. (One can imagine the gluing as being done in stages: first glue in a thickened $\mathcal{D}^2$ to get a boundary component that is an $\mathcal{S}^2$ and then glue in a final ball, which has no indeterminacy.) These are parameterized by the primitive (i.e. indivisible) elements of $H_1(\mathbb{T}^2) \approx \mathbb{Z}^2$.

Thurston's theorem now asserts that, if one excludes finitely many possibilities at each cusp, then all the remaining possibilities of filling produce hyperbolic manifolds. Moreover, as the boundary curves get longer and longer, the hyperbolic manifold that is constructed gets closer and closer to the original cusped hyperbolic manifold in a very reasonable geometric sense: The "surgery" can, up to very small perturbation, be imagined as taking place

[15] One needs to exclude phenomena analogous to companionship (which prevent any geometric structure) or torus knots (which correspond to structures that are not hyperbolic).

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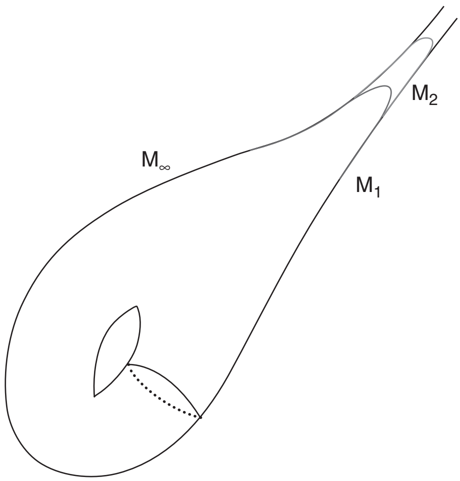

Figure 2.3 The limit of the volumes of the filled manifolds is the volume of the original cusped manifold. The cusped manifold is a pointed Gromov–Hausdorff limit of the filled manifolds.

further and further from the "core" of the original manifold.[16] (We will discuss the shape of noncompact locally symmetric manifolds at infinity more in Chapter 3.)

As a result, these manifolds have different volumes that converge to the volume of the original cusped manifold. This is a very crude reason for non-arithmeticity (although it does not do a single example!): for any G, the volumes of the arithmetic lattices $K\backslash G/\Gamma$ form a discrete subset of the positive reals.[17] Figure 2.3 gives a schematic of how the different "cusp closings" converge to the original cusped manifold.

It is very interesting to ponder this example from the representation theoretic viewpoint. One starts with a representation:

$$\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C})$$

that describes the original hyperbolic manifold with cusps. The filling gives nearby representations $\rho_n : \Gamma/\langle\gamma_n\rangle \to \mathrm{PSL}_2(\mathbb{C})$, where $\langle\gamma_n\rangle$ is the subgroup

[16] These examples provide a good set of examples for thinking about thick–thin decompositions, and the Cheeger–(Fukaya)–Gromov collapse theory.

[17] In some cases, they are even quantized (i.e. multiples of a given smallest one) using the Gauss–Bonnet theorem. I suspect that the converse holds, i.e. that only for Gs with $\chi(K\backslash G/\Gamma) \neq 0$ (this can be re-expressed in various ways – but, in particular for the case of hyperbolic manifolds, this is exactly that the dimension be even) are the volumes of torsion-free lattices quantized.

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normally generated by the nth filling curves. These provide a family of nearby but inequivalent representations to Γ.

Of course, Mostow's rigidity theorem asserts the uniqueness (up to conjugacy) of the discrete faithful representation. These representations are perturbations that are not faithful but are discrete. (For a closed manifold, all nearby representations to the discrete faithful one are in fact equivalent to it.) This phenomenon is highly special to this Lie group. Superrigidity is a significant strengthening of the representation-theoretic aspect of Mostow rigidity in high rank, and would preclude anything like this in higher rank.[18]

This method also has had a number of applications to constructing aspherical manifolds (and groups) that are not lattices. We will mention some in §2.3, and the method will recur when we discuss the groups of Gromov that disprove a version of the Baum–Connes conjecture[19] in Chapter 8.

2.2.3 Method Three: Gromov–Piatetski-Shapiro (G-PS) Grafting

This is the only method[20] that is known to produce examples in all dimensions. We describe the idea, but none of the technicalities, for which we refer to the original paper (Gromov and Piatetski-Shapiro, 1988).

Suppose that you have two compact arithmetic manifolds, and that they have a common codimension-1 submanifold. In other words, we have M and M' that are not (virtually) isometric, but both contain a separating totally geodesic submanifold V. Then we can cut both M and M' along V, and glue one side of M to the other side of M'. This is clearly a hyperbolic manifold.[21]

This manifold cannot be arithmetic, essentially because it has a big enough piece of M that it would have to be M if it were, but it would similarly have to be M', but it can't be both!

How do we get such pairs?

We get uniform lattices from orthogonal groups, but it is possible for different quadratic forms to give the same lattice. The condition is that the forms be similar (i.e. equivalent to rescaled versions of one another). Now it is pretty easy: If one takes the orthogonal groups of the quadratic forms ${x_1}^2 + {x_2}^2 + \cdots + {x_n}^2 - \sqrt{2}\,{x_{n+1}}^2$ and $3{x_1}^2 + {x_2}^2 + \cdots + {x_n}^2 - \sqrt{2}\,{x_{n+1}}^2$ over $\mathbb{Q}[\sqrt{2}]$, one gets noncommensurable lattices for n even. (The case of n odd is another trick away.) Now these each have an involution associated to $x_1 \to -x_1$ whose fixed

[18] Which is a good thing, because superrigidity gives rise to Margulis's arithmeticity (and therefore to discreteness of the set of volumes).

[19] This is a C^*-algebra version of the Borel conjecture.

[20] G-PS call it "interbreeding."

[21] More precisely, it is clearly a compact manifold with constant curvature equal to -1, but such are, of course, hyperbolic.

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set is a codimension-1 submanifold: essentially the orthogonal group of the lattice ${x_2}^2 + \cdots + {x_n}^2 - \sqrt{2}\,{x_{n+1}}^2$.

We will have use for the natural topological variant of this method for constructing interesting examples (such as counterexamples to certain orbifold variants of the Borel conjecture) in Chapter 7. One tries to find interesting aspherical objects with boundary and then obtains monsters by grafting[22] them together.

2.3 Some More Exotic Aspherical Manifolds

2.3.1 Method One: Davis's Reflection Group Method

This method was introduced by M. Davis (1983), whose self-proclaimed aim was to describe aspherical manifolds whose universal cover is not Euclidean space. There is a simple criterion (thanks to the Poincaré conjecture) for determining whether a contractible manifold is $\mathbb{R}^n$ or not; it is whether the manifold is *simply connected at infinity*.[23]

Recall that a manifold (or even locally compact space) is connected at infinity, if the complement of any compact subset has exactly one "noncompact" component (more precisely, one component with noncompact closure). Assume this is the case, then one can glob on all the compact components, to obtain a somewhat larger compact, whose complement has exactly one component.

Now let us consider a sequence of compact subsets that exhaust the space: $A_i \subset A_{i+1}$ and $M = \bigcup A_i$. The latter is simply connected at infinity if the inverse limit sequence

$$\pi_1(M - A_1) \leftarrow \pi_1(M - A_2) \leftarrow \cdots \leftarrow \pi_1(M - A_i) \leftarrow \cdots$$

is pro-equivalent to the trivial system, i.e. for each i, there is a $j > i$ so that $\pi_1(M - A_i) \leftarrow \pi_1(M - A_j)$ is trivial.

Note that this is *not* equivalent to there being "no loop that can be moved all the way to ∞." The system of $\mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \cdots$, where all arrows are multiplication by 2, has that property, but is not pro-trivial. The inverse limit is indeed trivial, but the multiples of 2^n come from n stages ahead – and this image does not stabilize.

At least for high-dimensional manifolds, this pro-triviality (i.e. simple connectivity at infinity) is equivalent to there being an exhaustion by compact sets, all of whose complements are simply connected. In general, the inverse limit

[22] Or interbreeding them.

[23] This criterion, in dimension > 4, is due to Stallings (1962) (and extended to dimension 4 by Freedman; in dimension 3, it follows from the Poincaré conjecture).

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of this sequence is independent of the defining compact sets, but this "fundamental group at ∞" only sometimes[24] plays the same role as the fundamental group for compact manifolds. In any case, it is a good approximation to "$\pi_1(\partial)$ if it only were the interior of a manifold with boundary ∂."

A good example of a contractible manifold (of dimension greater than 2) which is not Euclidean space is the interior of a contractible manifold whose boundary is non-simply connected. The boundary of a compact contractible manifold is automatically a homology sphere (i.e. has the homology of a sphere) – which is a sphere (according to the Poincaré conjecture) iff it is simply connected.

However, every homology sphere bounds a contractible topological manifold.[25] Some three-dimensional examples of homology spheres can be obtained by gluing together two nontrivial knot complements along their boundaries, interchanging longitudinal and meridional directions. Higher-dimensional examples can be obtained by spinning low-dimensional ones: puncture a homology sphere and cross it with a disk, and then take the boundary of this manifold.

Without relying on any theory, a simple example of a contractible 4-manifold whose boundary is non-simply connected is a Mazur manifold, constructed as follows: attach a $\mathcal{D}^2 \times \mathcal{D}^2$ to $\mathcal{S}^1 \times \mathcal{D}^3$ along a (neighborhood of a) nontrivial knot in $\partial(\mathcal{S}^1 \times \mathcal{D}^3) = \mathcal{S}^1 \times \mathcal{S}^2$ that represents a generator of $\pi_1 = \mathbb{Z}$.[26] (Mazur observed, see the crystal clear exposition in Zeeman (1962), that the product of this manifold with the interval $[0, 1]$ is a ball.)

Davis's idea was to generalize the obvious construction of $\mathbb{R}^2$ from a square by repeated reflection and gluing (producing the checkerboard with an action of the product of two infinite dihedral groups, $D_\infty \times D_\infty$) to a construction of some contractible manifold by reflecting across the top simplices of a triangulation of the boundary of any contractible manifold with boundary, with an action of a Coxeter group (that is, a group generated by reflections, whose only relations are commutation of the reflections along incident faces; see next page), whose quotient is precisely this "seed" contractible manifold.

Davis also calculated that, if the seed has non-simply connected boundary,

[24] Essentially when the manifold is tame at ∞.

[25] This is classical and due to Kervaire if the homology sphere is of dimension 4 and higher. It is strictly speaking correct in the PL and topological categories – in the smooth category it might be necessary to take the connected sum with an exotic sphere (a differentiable manifold homeomorphic to the sphere). In dimension 3 this is true in the topological category by the work of Freedman, but it is *not* true in the PL and smooth categories, by Rochlin's theorem, that the signature of a closed spin (smooth) 4-manifold is divisible by 16. The most straightforward proof of this important theorem is probably the one given in Lawson and Michelsohn (1989).

[26] Of course, to get an example, one should specify a knot and calculate that one gets a nontrivial homology sphere; but Gabai's theorem on "Property R" guarantees this.

then the manifold he so constructed is also non-simply connected at ∞. In particular, this happens if one starts with a Mazur manifold.

It is a general fact that Coxeter groups are linear, and therefore virtually torsion free, so a finite index subgroup acts on this contractible manifold freely, giving the relevant compact aspherical manifold with exotic universal cover.

Now for a few more details and a generalization with some indication of applications.

A right-angled[27] Coxeter group is given as a pair (Γ, V), where Γ is a group and V is a generating set, by elements of order 2. All the relations of Γ are consequences of relations of the form $(vw)^2 = 1$. We shall take the barycentric subdivision of a triangulation of our seed X. Define an abstract group Γ, generated by involutions v, one for each top simplex. We impose the relation $(vw)^2 = 1$ (and hence v and w commute) if the two simplices share a face. Note that if a k-tuple of simplices have pairwise commuting associated generators, then the intersection of these simplices is nonempty (and conversely). Consider $Z = \Gamma \times X/\sim$ where we identify points $(\gamma, x) = (\gamma', x')$ iff $\gamma^{-1}\gamma'$ lies in the group generated by all the generators of all the simplices that x lies in. So, in the interior of the seed, there is no identification. On the simplex corresponding to a generator v, v acts trivially. Davis proved by an induction on the length of the words in a Coxeter group that one obtains in this way a contractible manifold by showing that it is an ascending union of contractible spaces glued along contractible subspaces.[28]

The Davis construction is most usefully put into the context of CAT(0) geometry,[29] both in its own right in understanding the geometry that such a group has, and also because of the role that negative and non-positively curved geometry plays throughout our story. Nevertheless, we defer this discussion for now, and will say a bit more about it in describing the next construction.

Another variant that has extremely important applications is using strange seeds to construct aspherical manifolds with other strange properties. Start with any aspherical seed that is a manifold with boundary. Triangulating the boundary, and constructing the reflection group, one obtains here an aspherical manifold with a cocompact Coxeter action, and therefore, by passing to the universal cover, a contractible manifold with a cocompact group action, so on taking the quotient, a compact aspherical manifold which inherits properties from the seed. For example, this is a good way (following Davis and Haus-

[27] We assume right angles for simplicity. Otherwise, the exponent in the power for the nontrivial relations would be different.

[28] Perhaps reminiscent slightly of the argument in §2.1.

[29] CAT(0) is a synthetic notion of non-positive curvature, named by Gromov in honor of Cartan, Alexandrov, and Toponogov (see Gromov, 1987). Wait a page!

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mann, 1989) to produce an aspherical manifold with no smooth structure, or even no triangulation (using a seed that is a topological manifold that is non-triangulable, but whose boundary is triangulable, so that the construction can be done).

If K is any finite aspherical complex, one can take its regular neighborhood in Euclidean space[30] to obtain a manifold with boundary to use as a seed. This is a good way to produce aspherical manifolds whose fundamental groups are not residually finite or don't have a solvable word problem. An excellent book on Coxeter groups, their properties, and diversity is Davis (2008).

2.3.2 Method Two: Branched Covers (Gromov–Thurston Examples)

Gromov and Thurston (1987) gave some very interesting examples of compact manifolds with pinched negative curvature, i.e. curvature between -1 and $-1-\varepsilon$ by a variant of the philosophy of Dehn filling. This elaboration of that philosophy paves the way to other interesting constructions of groups by "adding large relations."

The basic idea is that negative curvature is a condition of large links. After all, negative curvature means that geodesics spread faster than in Euclidean space. So, if one takes a triangulated two-dimensional polyhedron, and then metrize it so that every triangle is an equilateral triangle with side length 1, then assuming that each vertex is incident to at least seven triangles should give a type of negative curvature. (As an exercise with Euler characteristic, neither the 2-sphere nor the torus has such a triangulation.)

A suitable version of curvature is given by the notion of CAT(k) geometry. A metric space X is called geodesic if its metric is generated by the length of paths connecting pairs of points. Riemannian manifolds are a good example, but one can make others by using a metric and then taking lengths of paths. Now, suppose that we have a triangle in X; then we can construct a triangle in one of the model geometries with curvature k (i.e. rescaled hyperbolic space, Euclidean space, or a sphere). We say X is CAT(k) if the triangles in X are thinner than the corresponding model triangle, meaning that each leg is closer to the union of the other two in X than they are in the model. Figure 2.4 shows a δ-thin triangle.

This is equivalent to curvature less than k for Riemannian manifolds and is

[30] Any (finite) polyhedron can be simplicially embedded by general position in a much larger-dimensional Euclidean space. Subdividing, and taking the union of all of the simplices that touch this complex, one obtains a (compact) manifold with boundary that (simplicially collapses onto and therefore) deformation retracts to the polyhedron. This is called a *regular neighborhood*.

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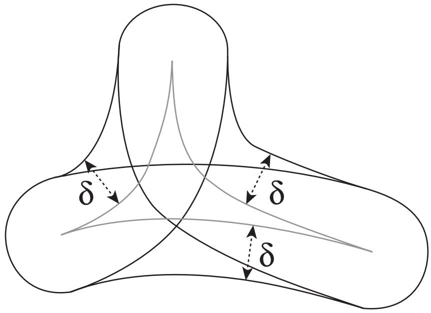

Figure 2.4 A δ-thin triangle.

a useful synthetic substitute for other metric spaces. If X is locally CAT(0), then its universal cover is contractible. (Points will be connected by unique geodesics, and the contraction will be radial.) A great example is a tree.

Back to the case of triangulated surfaces: six incident triangles for each point implies CAT(0), seven gives a negative[31] CAT curvature.

In the Dehn surgery theorem, we can think of the process of filling as gluing on (a family of) $\mathcal{D}^2$s along the translates of a geodesic on the boundary torus. Thurston's theorem tells us that we can have negative curvature (indeed, he gives constant, but that's too much in general) if the length of the geodesic is long enough.

Gromov and Thurston do something similar. They consider a hyperbolic manifold M with a totally geodesic submanifold V of codimension 2. They show that k-fold branched covers[32] (can be proved to exist, at least sometimes, and then) can be given metrics with curvature between -1 and $-1 - c/\log(k)$. The volume in this construction grows linearly: the metric is constructed quite explicitly and deviates from the hyperbolic metric only in a small neighborhood of the submanifold (as the heuristic suggests).

Philosophically, when k gets large, the curvature should be getting more negative. They essentially have to stretch the neighborhood to make it more pinched (i.e. so that the divergence of the geodesics has more time to occur).

The reason that these manifolds can't be made constant negative curvature is a nice application of Mostow rigidity. They all have $\mathbb{Z}_k$-actions, which would be isometric if they were constant curvature. Varying k and modding out by

[31] Depending on the length of the triangles.

[32] Recall that a branched cover of a manifold M along a codimension-2 submanifold V is a cyclic covering space of the complement $M - V$ that restricts to the usual cyclic cover of the circle to itself in the direction normal to V. This allows one to fill in V in the covering space, and obtain a manifold (with $\mathbb{Z}_k$-action – whose fixed set is V, and whose quotient is M).

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the actions would produce infinitely many different hyperbolic orbifolds with bounded volume. However, above dimension 3, there are only finitely many hyperbolic orbifolds with any given volume bound (Wang's theorem: see Wang, 1972).

As the final technical point to mention, one can take branched cover along a codimension-2 submanifold iff it is trivial as a homology class. We can construct examples of this by the arithmetic construction we discussed earlier. If one uses the quadratic forms that arose in "grafting," then there is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action generated by two reflections. The fixed set of the action of the whole group is null-homologous in the fixed set of either of the involutions – which gives us the relevant M and V to start this construction.

In general, this method is about adding long relations and keeping negative curvature. This method is related to the ideas of small cancellation theory (as is CAT(-1) geometry in general) and in both its manifold and nonmanifold versions has led to many very interesting groups, some of which we will discuss below.

2.3.3 Method Three: Hyperbolization

The basic idea of hyperbolization is very simple, and there are many hyperbolization methods, i.e. implementations of this idea. We will be brief and leave the reader to study the (rather beautiful) literature (see the notes in §2.4). On the other hand, it is impossible to resist mentioning at least a few of the surprising examples.

The Kan–Thurston theorem asserts that any simplicial complex X has the homology type of a group π, i.e. there is a map $\mathrm{B}\pi \to X$ which is an isomorphism (for all local coefficient systems on X). Baumslag, Dyer, and Heller gave a very nice approach to this theorem that gives a finite complex $B\pi$ if X is finite (Baumslag *et al.*, 1980).

The idea is to find a "simplex of acyclic groups" and glue these together. One simple version can be done as follows, using cubes instead of simplices. This doesn't make a difference since one can replace every simplicial complex by a "cubulated" complex. So we will instead look for cubes of acyclic groups.

Acyclic groups are easy to come by. A simple example is any free product with amalgamation $\pi = F *^{F'} F$, where F and F' are free groups of rank k and $2k$, respectively, and the first inclusion of F' to F induces a split surjection on first homology $\mathbb{Z}^{2k} \to \mathbb{Z}^k$ projecting onto the first k dimensions, and the second inclusion interchanging the first k and second k basis elements. In this case, gluing tells us that $\mathrm{B}\pi$ is the 2-complex obtained by taking the double

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mapping cylinder of a wedge of $2k$ circles mapping to two wedges of k circles. A straightforward Mayer–Vietoris calculation then gives that $B\pi$ is acyclic.

An "interval of acyclic groups" is simply given by the diagram of groups $\pi \to \pi \times \pi \leftarrow \pi$, where the first (respectively second) inclusion is given by the inclusion of the first (second) factor. From intervals of groups, we can obtain squares and the cubes of groups by taking products.

Note that if we have a cubical complex, we can then (ordering the vertices!) glue together the associated cube of acyclic groups. This will produce a complex (in this case, finite if X is, and of twice the dimension) which has all the desired properties.

Notice that this construction is aspherical in the category of simplicial complexes (or cubical complexes) and simplicial inclusions.

Hyperbolizations do exactly the same thing, but using aspherical *manifolds*[33] instead of complexes. In both of these constructions, it is critically important that fundamental groups inject for gluing purposes.

It is not possible to arrange for the map to be a homology equivalence (for then the 2-sphere would be homology-equivalent to an aspherical surface – which we know by classification is not the case).[34] However, other geometric properties can be achieved by suitable constructions of simplices or cubes of aspherical manifolds.

The seed is often chosen to be non-positively curved (or negatively curved), orientable, or even with stably trivial[35] tangent bundle. Points are hyperbolized as points, and the geometry is rigid enough that the links of these points are the same in X and its hyperbolized version. If X is a manifold, so will be the hyperbolized space, and the map $\mathbb{H}(X) \to X$ will be degree 1 and preserve characteristic classes. This implies that $\mathbb{H}(X)$ is cobordant to X,[36] so, for example, every cobordism class contains an aspherical manifold.

If M is a manifold with boundary, Gromov suggested hyperbolizing $M \cup c\partial M$ (where $c\partial M$ denotes the cone of the boundary of M). This will produce an aspherical complex with a single singular point, whose link is ∂M. One can show that if ∂M is aspherical, then one can remove this singular point to get a "relative hyperbolization" that ∂M bounds (mapping to M). Thus, not only is every manifold cobordant to an aspherical manifold, but also cobordant aspherical manifolds are cobordant through aspherical manifolds.

33 We give up on acyclicity, however.

34 In higher dimensions, I do not know how to eliminate any closed manifold from being homology-equivalent to an aspherical manifold. However, the Hopf conjecture would clearly preclude this.

35 That is, trivial after adding on a trivial bundle.

36 By Thom's classical work that shows that bordism is governed by tangential information (Thom, 1954).

Among the applications of this technique (besides ones we will see later) are aspherical manifolds that cannot be triangulated or smooth manifolds (whose universal covers are topologically $\mathbb{R}^n$) with CAT(0) metrics, but no Riemannian metric of non-positive curvature.

For example, a non-triangulable aspherical manifold comes from the following. The Poincaré homology 3-sphere[37] Σ bounds a 4-manifold W whose intersection form is E_8 (the unique eight-dimensional positive definite unimodular quadratic form over $\mathbb{Z}$ with $\langle x, x\rangle \equiv 0 \bmod 2$ for all x).[38] Hyperbolize $W \cup c\partial$. Then remove the cone point, and glue on the contractible 4-manifold, constructed by Freedman, that Σ bounds. This gives a topological manifold X that, being homotopy equivalent to the hyperbolization, is aspherical. On the other hand, this manifold is "spin" in the sense that its first two Stiefel–Whitney classes must vanish (since they do for W, and hyperbolization is tangential), which then prevents smoothness – by the cobordism property X has signature 8, but Rochlin's theorem asserts that any smooth spin 4-manifold has signature a multiple of 16.

The complex X cannot be triangulated as a simplicial complex, as can be seen using either the Casson invariant[39] (or, even easier now, the three-dimensional Poincaré conjecture).

Ontaneda (2011) refined the construction of hyperbolization to produce arbitrarily well-pinched negatively curved hyperbolizations, so one can, for instance, construct manifolds with curvature $-1 - \varepsilon < k < -1$ in any cobordism class.

2.4 Notes

That surfaces tend to be aspherical is classical. For 3-manifolds, there were some early results by combinatorial methods. For example, Aumann (1956)[40] proved the result asserted in its title with its main topological tool being the gluing lemma. That 3-manifolds in general tend to be aspherical (and, for example, the complements of all knots, and all nonsplittable links) is due to Papakyriakopoulos (1957).

The tools introduced in that paper (the Dehn lemma, loop, and sphere the-

[37] See Kirby and Scharleman's (1979) for a beautiful description of this 3-manifold and many descriptions and properties of it.

[38] See Serre (1973) for more information.

[39] Casson showed how to count the conjugacy classes of SU(2) representations of the fundamental group of the homology 3-sphere, and that, when done properly, these reduce mod 2 to $1/8$ of the signature of any smooth cobounding spin 4-manifold.

[40] Whose author later won a Nobel Prize (for work in game theory).

orems) were the core of 3-manifold topology (their power being most evident for the class of “Haken” 3-manifolds) until the Thurston revolution brought in a wealth of more (differential) geometric techniques. This development can be found in any standard book on 3-manifolds. (Good books for the torus decomposition and some of its pre-Thurston understanding are Hempel, 1976, and Jaco, 1980.)

The geometricization conjecture of Thurston is a picture of *all* closed 3-manifolds in terms of locally symmetric ones. The possible geometries are well described in Scott (1983). Very useful explanations of Thurston’s theorem proving this picture correct in the situation where there is an incompressible surface are Morgan (1984) and Kapovich (2001) (from a different point of view than Thurston’s original approach). A detailed explanation of Perelman’s result can be found in Morgan and Tian (2014).

The study of locally symmetric manifolds started in the nineteenth century. These manifolds are now studied by mathematicians of many different stripes. Besides being interesting examples to geometers, the geometry and topology of many of these manifolds are the essence of such classical results of algebraic number theory as the Dirichlet unit theorem (which calculates the group of units in the integers of an algebraic extension of $\mathbb{Q}$, and which is the compactness of a certain torus) and the finiteness of the class number (which, for instance in the situation of a totally real field, follows from the existence of a compactification for Hilbert modular varieties – the cusps corresponding to elements of the class group). We will discuss arithmetic manifolds and hints of arithmeticity in Chapter 3. As mentioned earlier, Borel (1963) gave the first general construction of uniform lattices for all $K\backslash G$. It is much simpler to give non-uniform lattices. The books by Eberlein (1997) and Witte-Morris (2015) are extremely useful.

Non-arithmetic lattices, as we have seen, are ubiquitous (if not so easy to construct) in low dimensions. The question of exactly which semisimple Lie groups admit them is still open. As we mentioned, for rank greater than 1, Margulis’s arithmeticity theorem assures us that there are no (irreducible) examples (see Zimmer, 1984; Margulis, 1991).

The only known construction that works in infinitely many dimensions is the Gromov–Piatetski-Shapiro (G-PS) grafting method we explained. Deligne and Mostow (1993) gave some examples in $\mathrm{U}(n, 1)$ for small n. On the other hand, in $\mathrm{Sp}(n, 1)$ and F_4, Gromov and Schoen (following on earlier work of Corlette) showed that arithmeticity does hold using analytic methods related to harmonic maps (Gromov and Schoen, 1992).

The G-PS manifolds play a role in counting the number of hyperbolic manifolds with volume less than V, in dimensions greater than 3 (when it is finite)

(Burger *et al.*, 2002) and with diameter less than D in all dimensions, including dimension 3 (Young, 2005).[41]

As emphasized in the main text, the examples of non-arithmetic lattices are suggestive of tools for constructing interesting aspherical manifolds that have nothing to do with lattices. Davis (1983, 2000) was motivated, as he explains therein, by Andreev's theorem about reflection groups in hyperbolic space.

Closing cusps has been applied both to manifolds and to nonmanifolds. See Hummel and Schroeder (1996) for the situation of closing cusps for, e.g., complex hyperbolic manifolds (and its impossibility in the quaternionic case). CAT(0) geometry was broadcast to the world by Gromov (1987) in his paper on hyperbolic groups. The main theme of that paper is developing a large-scale (or coarse) notion of negative curvature for groups, as a property of their Cayley graphs, and showing how this notion deepens and generalizes our understanding of hyperbolic manifolds. The most obvious examples of such groups are fundamental groups of negatively curved manifolds, and also free groups. But there are many more!

For instance, Gromov pointed out that one can cone very long words at will[42] (as a generalization of the idea of Thurston's Dehn surgery theorem) and maintain negative curvature, giving "easy" finitely generated torsion groups (just kill large powers of the elements of the group, one at a time).[43]

That paper also introduces hyperbolization (with some glitches regarding the procedure fixed in Davis and Januszkiewicz, 1991, Charney and Davis, 1995, and Davis *et al.*, 2001), which also give some new applications. The paper, all told, launched a major area of geometric group theory and numerous other investigations. See, for example, Ghys and de la Harpe (1990) for an exposition of much of the content of that paper. Bridson and Haefliger (1999) is an excellent source on non-positively curved spaces that are not necessarily manifolds.

Regarding more basic facts about discrete groups that arose in this chapter, see C. Miller (1971) for constructions of groups with unsolvable word problem and related matters. Baumslag *et al.* (1980) is the paper that gives the finite form

[41] Note that when a hyperbolic Dehn surgery is done, the filling takes place further and further down the cusp, and the diameter of the manifold increases with the length of the curve filled.

[42] What I mean is that one can represent a long word in $\pi_1 X$ by a long closed geodesic in X, and then we can attach a disk along this word, and maintain negative curvature. If the geodesic is long, then the geometry is that of locally having an n-gon with $n > 6$ at the new vertex.

[43] If one starts with a lattice in $\mathrm{Sp}(n, 1)$ and does this, one gets an infinite torsion group with Property (T). This example also shows that while Property (T) implies finite generation, it does not imply finite presentation. (See Chapter 3 for the basics of Property (T).) On the other hand, this method does not solve the Burnside problem of giving finitely generated exponent p groups. However, even this can be achieved in the hyperbolic group setting, as was shown by Ivanov and Ol'shanskii (1996).

of the Kan–Thurston theorem along the lines described here. It is subsequently applied in Baumslag *et al.* (1983) to give remarkable information about the possible sequences of homology groups of a finitely presented group (it's obviously not arbitrary: there are countably many finitely presented groups and uncountably many sequences of even finite abelian groups!).[44]

Rochlin's theorem, mentioned in explaining the construction of a non-triangulable four-dimensional aspherical manifold, asserts that the signature (see Chapter 4) of a smooth spin 4-manifold is a multiple of 16. This was immediately understood to be an anomaly, and led to various examples of phenomena where dimension four behaves differently from the smooth perspective than higher dimensions. This turned out to be the tip of the iceberg with the advent of Donaldson's thesis (see Donaldson, 1983) – and the work that has followed it – which has yielded much more profound information about smooth 4-manifolds.

[44] It also contains the construction of an acyclic universal group, i.e. an acyclic finitely presented group containing every finitely presented group as a subgroup. (Note that there's no finitely generated group containing all finitely generated groups.) This group has been surprisingly helpful for various constructions. As one example relevant to this chapter, it was applied in an early version of Davis *et al.* (2001) for the construction of relative hyperbolization – although this was not necessary in the final version, which followed Gromov's original ideas more closely.