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THE LARGE STRUCTURES OF GROTHENDIECK FOUNDED ON FINITE-ORDER ARITHMETIC

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Abstract. The large-structure tools of cohomology including toposes and derived categories stay close to arithmetic in practice, yet published foundations for them go beyond ZFC in logical strength. We reduce the gap by founding all the theorems of Grothendieck's SGA, plus derived categories, at the level of Finite-Order Arithmetic, far below ZFC. This is the weakest possible foundation for the large-structure tools because one elementary topos of sets with infinity is already this strong.

CONTENTS

§1. Outline. Grothendieck's number theory links large structures to small. Notably, each single scheme has a large category of sheaves. The point is not to study vastly many sheaves but to give a unifying framework for general theorems. Grothendieck gave a set theoretic foundation using *universes*, which he described informally as sets "large enough that the habitual operations of set theory do not go outside" them (SGA 1 VI.1 p. 146). Some authors avoid the large structures, at least officially, because Zermelo Fraenkel set theory with choice (ZFC) cannot prove these universes exist. But the large structures reappear in citations and as motivation. This article removes the objection by founding Grothendieck's large tools on a fragment of ZFC with the logical strength of Finite-Order Arithmetic.

Finite-Order Arithmetic (Takeuti, 1987, Part II), or Simple Type Theory with infinity, axiomatizes the theory of numbers, sets of numbers, and sets of those, up through any finite level. §2–§3 present a set theory with this consistency strength and show it proves the basic theorems of derived functor cohomology.

§4 gives a higher order theory conservative over that set theory, which §5–§7 use to formalize the large structures, including derived categories and a 2-category of Grothendieck toposes, little changed from Grothendieck's original. This is the lowest possible

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consistency strength for Grothendieck's tools since one elementary topos of sets with infinity is already this strong.[1]

1.1. First technical key. The first technical key here is: no theorem in Grothendieck's *Elements of Algebraic Geometry* (EGA) or *Seminar on Algebraic Geometry* (SGA), outside of the appendix on set theory in SGA 4, needs either the unbounded axiom scheme of separation or the axiom scheme of replacement for its proof. §2.3 discusses those unneeded axioms.

Most of the thousands of pages of EGA and SGA deal with elementary algebra of countable rings and modules. There is occasional use of continuum-sized fields like $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$. None of this uses replacement or unbounded separation.

Grothendieck did use replacement, though. His Thm. 1.10.1 of (1957a) is fundamental to all the cohomology in EGA and SGA, and his proof invokes replacement.[2] Beyond that, replacement has a role in Grothendieck's idea of *derived categories*, see §7.5. There is no principled reason he might not have invoked replacement in other ways, but he did not and neither have later geometers pursuing these ideas. §3.8 shows in detail that bounded separation suffices for Grothendieck's Theorem 1.10.1. §7.5 looks at replacement in derived categories.

Grothendieck, like most mathematicians, habitually gives quantifier bounds for separation. But not always. We fill in missing bounds for paradigmatic cases.

1.2. Second technical key. Gödel-Bernays set theory (GB) extends ZFC by adding proper classes of sets, yet GB is conservative over ZFC because formulas defining sets or classes in GB can only quantify over sets and not classes. Our class theory is conservative over our set theory in that same way, and suffices to prove the large structure theorems of EGA and SGA plus duality. In terms of logic, this works because the relevant definitions of classes and collections only quantify over sets, and then only sets with bounded definitions. In categorical terms, that translates to saying these large structures are *locally small* (Definition 6.1).

§2. Suitable set theories.

Many set theories suit our purposes. All have the same consistency strength. Relevant ones are: Finite-Order Arithmetic, the Elementary Theory of the Category of Sets (ETCS) (Lawvere, 1965), and various fragments of ZFC. To facilitate comparison with ZFC we use a fragment of it, called MacSet for "Mac Lane set theory." It has these axioms (Mac Lane & Moerdijk, 1992, p. 332):

- *Extensionality*: $x = y$ if, for all t, $t \in x$ iff $t \in y$.
- *Null Set*: There is a set $\emptyset$ with $x \notin \emptyset$ for all x.
- *Pair*: For all sets x and y there is a set $\{x, y\}$.
- *Union*: For all x there exists $\cup x$ with $t \in \cup x$ iff some $y \in x$ has $t \in y$.
- *Power Set*: For all x there exists $\mathcal{P}(x)$ with $t \in \mathcal{P}(x)$ iff $t \subseteq x$.
- *Infinity*: There exists a set of all natural numbers $\mathbb{N}$.
- *Choice*: If x is a set with $y \neq \emptyset$ for all $y \in x$ then there exists a function f on x with $f(y) \in y$ for all $y \in x$.

[1] Experts will see much of the argument is indexed category theory over a well-pointed topos. For this special case, indexing just amounts to local smallness as in §6–§7.

[2] Grothendieck's Theorem 1.10.1 shows suitable categories have "enough injectives."

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Bounded Separation: For any formula $\psi(x)$ with all quantifiers *bounded*, and any set y, there is a set $\{x \in y \mid \psi(x)\}$.

A *bounded* quantifier has the form $\forall x \in \tau$ or $\exists x \in \tau$ where the *bound* τ is some term indicating a set.[3] This is quite usual in mathematics. A number theorist will write $\exists x \in \mathbb{Z}[\sqrt{-1}] \ldots$ to say "for some Gaussian integer x...." Or a geometer will write $\forall p \in M \times N \ldots$ to say "for every point p of the product manifold $M \times N$ ...." Bounded quantifiers do not refer to some or all sets; they refer to some or all members of a given set τ.

Crucially, this separation axiom uses both a bounded defining property $\psi(x)$ and an *ambient set* y to define a set. The defined set is a subset of the ambient:

$$\{x \in y \mid \psi(x)\} \subseteq y.$$

The same proofs work in MacSet as in ZFC to show every pair of sets A, B has a binary union $A \cup B$ and a cartesian product $A \times B$. We choose the usual representative for the pullback of functions $f : A \to C$ and $g : B \to C$:

$$A \times_C B = \{\langle x, y\rangle \in A \times B | f(x) = g(y)\}. \tag{1}$$

Formally, a function $f : A \to B$ is an ordered triple $\langle A, B, R\rangle$ with any sets A, B, and $R \subseteq A \times B$ a *functional relation* from A to B. That is, every $x \in A$ has a unique $y \in B$ with $\langle x, y\rangle \in R$. Then R is the *graph* of f and may be called Γ_f. By the usual proof, MacSet shows sets A, B always have a function set B^A.

Unlike in ZFC, not every expression $\psi(x, y)$ relating each $x \in A$ to a unique $y \in B$ defines a function $f : A \to B$. In MacSet, $\psi(x, y)$ must have all quantifiers bounded to define a subset of $A \times B$, and thus define a function.

2.1. Indexed sets. Although everything in MacSet is a set of sets, explicitly indexed sets of sets $\{X_i | i \in I\}$ are indispensable to us. Such a set cannot be proven to exist in MacSet merely by defining a set X_i for each $i \in I$. In MacSet (unlike ZFC) a selection of sets X_i can be definable while no ambient set X contains all the X_i. Then MacSet cannot collect the X_i into one set.

So we define an I-indexed set of sets $\{X_i | i \in I\}$ to be a function $s : X \to I$ to I from some single ambient set X. Each $X_i \subseteq X$ is the part of X lying over index $i \in I$. So it is defined by

$$X_i = \{x \in X | s(x) = i\} \subseteq X.$$

Up to a canonical natural isomorphism, X_i is the pullback of s along the obvious $i : 1 \to I$.

So X is the disjoint union $\coprod X_i$ of the X_i. And the product $\prod_i X_i$ is the set of all $f : I \to X$ such that $sf = 1_X$.

Indexed sets have this relation to arbitrary sets of sets:

LEMMA 2.1. *Every set S of disjoint sets appears as an indexed set as $s : \cup S \to S$ where $s(y) = x$ for each $y \in x \in S$. And every set S is naturally isomorphic to a set of disjoint sets in many ways, for example the obvious explicitly defined isomorphism of S to the set $\{x \times \{x\} | x \in S\}$.*

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3 It makes no difference whether we require τ to be a variable or allow using provably well-defined functions like power set or bounded set abstracts, so long as x is not free in τ.

2.2. Rank over a set. In MacSet as in ZFC there is no set of all groups, but there is a set of all groups with underlying set G for any given set G. This is more or less obvious and a correct generalization of it is central to both Grothendieck's and our uses of $\mathcal{U}$-categories. The present section does for us what the appendix on set theory to SGA 4 exp. I does for Grothendieck's approach.

Groups make a good example: Define a group set theoretically as an ordered pair $\langle G, m\rangle$ of a set G and a "multiplication table" $m \subseteq (G \times G) \times G$ satisfying the group axioms. If we use the Kuratowski pairing $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$, then m is a subset of $\mathcal{P}^2(\mathcal{P}^2(G) \cup G)$, where $\mathcal{P}^2$ indicates the power set of the power set. Altogether, the group is an element of this iterated sum and power set:

$$\langle G, m\rangle \in \mathcal{P}^2\Big(G \cup \mathcal{P}^3\big(\mathcal{P}^2(G) \cup G\big)\Big).$$

Indeed every group with underlying set contained in G occurs as an element of that set. So bounded separation yields a set of all groups with underlying set G.

Now, for each set S define a finitary cumulative hierarchy:

$$V_0(S) = S \quad \text{and} \quad V_{n+1}(S) = V_n(S) \cup \mathcal{P}(V_n(S)).$$

DEFINITION 2.2. *Call* $V_n(S)$ *the set of* sets of rank at most n *over* S.

In these terms every group $\langle G, m\rangle$ has rank at most 7 over the set G. And every group homomorphism $h\colon G \to H$ has rank at most 11 over the union $G \cup H$ of the underlying sets. Those exact numbers 7 and 11 are sensitive to details of coding groups and homomorphisms, but what matters is that any reasonable coding will make both of the rank bounds finite and calculable.

THEOREM 2.3. *For each given natural number n, MacSet proves the quantified statement*: $\forall S\, \exists y\, (y = V_n(S))$.

Proof. Straightforward by the power set, pair set, and union axioms. □

MacSet proves this with a quantifier $\forall S$ over sets, but only for each single natural number n. MacSet cannot prove it with a quantifier $\forall n \in \mathbb{N}$ over natural numbers, for reasons sketched in §2.3.

Corresponding to SGA 4 exp. I Proposition 4 Corollary 1 we have

COROLLARY 2.4. *Consider any species of structure definable by a multisorted n-th order theory, or what is nearly the same consider models of any of Bourbaki's échelles de structure. Let the morphisms be* (*possibly equivalence classes of, possibly partially defined*) *functions between the sorts which preserve and/or reflect various aspects of the structure. Examples could be Lie groups, or measure spaces and measurable functions. Then for any fixed set G, MacSet proves there is a set of all structures of that species where the sorts are all interpreted by subsets of G*; *and there is a set of all morphisms between those.*

Proof. For each given species of this kind, and set G, the desired sets will be elements of some $V_m(G)$, defined by relations among elements of $V_m(G)$. □

2.3. MacSet, Zermelo, and Zermelo-Fraenkel set theories. This section is not used in what follows, but may help in understanding MacSet.

The point is to distinguish *defining* a set from *proving it exists*. All three of these set theories can define, for example, the finitely iterated power sets of $\mathbb{N}$ and the set of all of

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them:

$$\mathcal{P}^0(\mathbb{N}) = \mathbb{N}, \quad \mathcal{P}^{n+1}(\mathbb{N}) \ = \mathcal{P}(\mathcal{P}^n(\mathbb{N})), \ \text{and} \ \{\mathcal{P}^n(\mathbb{N}) | n \in \mathbb{N}\}.$$

The *axiom scheme of replacement* in Zermelo-Fraenkel set theory (ZFC) proves all these sets exist. Then it proves there exists a yet-larger power set of the union of all $\mathcal{P}^n(\mathbb{N})$, and far more. This axiom scheme is so much stronger than MacSet that we will not state it but merely refer to any account of ZFC.

Instead of replacement, Zermelo set theory has the *unbounded* axiom scheme of separation: for each formula $\psi(x)$ and set y there is a set $\{x \in y | \psi(x)\}$. For example let $\psi(n)$ be this formula where n is a free variable:

$$\text{there exists an } n\text{-th power set } \mathcal{P}^n(\mathbb{N}). \tag{2}$$

The unbounded separation axiom affirms there is a set:

$$\{n \in \mathbb{N} | \text{ there exists an } n\text{-th power set } \mathcal{P}^n(\mathbb{N})\}. \tag{3}$$

Induction on n shows this set is all of $\mathbb{N}$. In other words, Zermelo set theory proves all the finitely iterated power sets exist:

$$\forall n \in \mathbb{N} \text{ there exists } \mathcal{P}^n(\mathbb{N}). \tag{4}$$

But Zermelo set theory cannot prove $\{\mathcal{P}^n(\mathbb{N}) | n \in \mathbb{N}\}$ exists, since that plus the axioms of Zermelo set theory would provide a model of Zermelo set theory.

MacSet has power sets so it proves each specific *iterated power* set of $\mathbb{N}$ exists, such as $\mathcal{P}^3(\mathbb{N})$ or $\mathcal{P}^{42}(\mathbb{N})$. But it cannot prove the quantified statement (4) about all $\mathcal{P}^n(\mathbb{N})$. It cannot use induction on formula (2), because it cannot prove formula (2) defines a set! The far from obvious proof is in Mathias (2001).

§3. Basic cohomology in MacSet. The sections below often cite published proofs and claim they work in MacSet, without repeating them. The cited proofs rest on explicit constructions.[4] We only need to assure the constructions can be done with bounded separation. This is done very fully for Theorem 3.2 as an example. Ambient sets for the crucial Theorem 3.14 depend on substantial prior results.

3.1. Small categories. A *small category* $\mathbf{C}$ is a set C_0 called the set of objects and a set C_1 called the set of arrows with domain and codomain functions d_0, d_1, identity function id, and composition m satisfying the category axioms. Set theoretically $\mathbf{C}$ is an ordered 6-tuple $\langle C_0, C_1, d_0, d_1, id, m\rangle$.

A *functor* $f : \mathbf{C} \to \mathbf{D}$ of small categories is an ordered pair $\langle f_0, f_1\rangle$ of an *object part* $f_0 : C_0 \to D_0$ and *arrow part* $f_1 : C_1 \to D_1$ preserving domains, codomains, composition, and identity. After §5 we may call these *small functors*.

For small categories $\mathbf{B}, \mathbf{C}$ the ordinary textbook proof that there is a set of all functors $\mathbf{B} \to \mathbf{C}$ works verbatim in MacSet. The following stronger result takes slightly more care:

DEFINITION 3.1. *An I-indexed set of small categories $\{\mathbf{C}_i | i \in I\}$ consists of an I-indexed set of sets $C_0 \to I$ taken as the set of sets of objects $\{C_{0i} | i \in I\}$ and another $C_1 \to I$ as the set of sets of arrows $\{C_{1i} | i \in I\}$; plus suitable domain, codomain, and composition functions between them.*

[4] McLarty (2006) discusses the relatively constructive aspect of category theory.

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THEOREM 3.2. *For any I-indexed set of small categories $\{\mathbf{C}_i | i \in I\}$ there is an $I \times I$-indexed set of all functors between them: $\{f : \mathbf{C}_i \to \mathbf{C}_j | \langle i, j\rangle \in I \times I\}$.*

First, Hands-on Proof. The strategy is to find an ambient set Y containing all the functors between any of these categories, then give quantifier bounds for the formula defining the functors within Y. Consider these cartesian products:

(1) $\mathcal{P}(C_0) \times \mathcal{P}(C_0) \times \mathcal{P}(C_0 \times C_0)$
(2) $\mathcal{P}(C_1) \times \mathcal{P}(C_1) \times \mathcal{P}(C_1 \times C_1)$
(3) $\Big(\mathcal{P}(C_0) \times \mathcal{P}(C_0) \times \mathcal{P}(C_0 \times C_0)\Big) \times \Big(\mathcal{P}(C_1) \times \mathcal{P}(C_1) \times \mathcal{P}(C_1 \times C_1)\Big)$

For fixed i, j the graph of the object part of any functor $\mathbf{C}_i \to \mathbf{C}_j$ is a subset of $C_{0i} \times C_{0j}$ and thus a subset of $C_0 \times C_0$. But $C_0 \times C_0$ works for every pair i, j at once. So the power set $\mathcal{P}(C_0 \times C_0)$ contains as elements the graphs of the object parts of all the functors we want plus a lot of junk. Thus, to specify the domains and codomains, the triple product labelled (1) contains the object parts of all the functors, plus junk. Similarly the product (2) contains the arrow parts of all the functors. All the functors are elements of product (3). That is our ambient set.

The functors within the ambient set (3) are defined by equations on the domain, codomain, identity, and composition functions of the categories $\mathbf{C}_i$, quantifying only over elements of I, C_0, C_1 and ordered pairs or triples of those elements. □

Second, Proof by Rank. Since the $I \times I$-indexed set of functors is constructed from C_0 and C_1 by a fixed finite number of power sets and products, it has rank below some number n over the union $C_0 \cup C_1$. So some $V_n(C_0 \cup C_1)$ suffices as ambient by Theorem 2.3. The desired indexed set is defined within that ambient by equations quantifying only over I, C_0, C_1 and finite products of those. □

The first proof shows $n = 7$ suffices for the second proof. But the second proof does not require knowing n.

This theorem says the category $\mathcal{C}at$ of all small categories in MacSet is locally small, except that $\mathcal{C}at$ is a proper class and does not exist in MacSet.

3.2. Diagrams and presheaves. Intuitively a diagram of sets on a small category $\mathbf{C}$ is a covariant set-valued functor $\mathcal{F}: \mathbf{C} \to \mathcal{S}et$, and.a presheaf on a small category $\mathbf{C}$ is a contravariant set-valued functor $\mathcal{F}: \mathbf{C}^{op} \to \mathcal{S}et$. §6.1 formalizes presheaves that way using class theory. For most applications, though, the more fruitful viewpoint has been the *Grothendieck construction* defining diagrams and presheaves without using $\mathcal{S}et$ or other proper classes (see *internal diagrams* in Mac Lane & Moerdijk, 1992, p. 243). §6 and §7 will deal with proper classes, and so will sometimes refer to diagrams and presheaves defined within MacSet as *small*.

We will present presheaves in detail, and merely define a diagram of sets on a small category $\mathbf{C}$ as a presheaf on the opposite category $\mathbf{C}^{op}$.

A *presheaf* $\mathcal{F}$ on a small category $\mathbf{C}$ is a C_0-indexed set $\gamma_0 : F_0 \to C_0$ called the *set of sections*; and a function $e_{\mathcal{F}} : F_1 \to F_0$ called the *action*, where

$$F_1 \;=\; F_0 \times_{C_0} C_1 \;=\; \{\langle s, f\rangle \in F_0 \times C_1 \mid \gamma_0(s) = d_1(f)\}.$$

For each $A \in C_0$, the value $\mathcal{F}(A)$ is called the set of *sections over* A:

$$\mathcal{F}(A) = \{s \in F_0 \mid \gamma_0(s) = A\}.$$

The action is required to satisfy axioms:

(1) For all arrows $g\colon A \to B$ in $\mathbf{C}$, if $s \in \mathcal{F}(B)$ then $e_{\mathcal{F}}\langle s, g\rangle \in \mathcal{F}(A)$.
(2) If $s \in \mathcal{F}(A)$ then $e_{\mathcal{F}}\langle s, 1_A\rangle = s$ for the identity arrow 1_A.
(3) For any $h\colon C \to B$ in $\mathbf{C}$, $e_{\mathcal{F}}\langle s, gh\rangle = e_{\mathcal{F}}\langle e_{\mathcal{F}}\langle s, g\rangle, h\rangle$.

By Clause 1, the action $e_{\mathcal{F}}$ applied to an arrow $g\colon A \to B$ defines a function $\mathcal{F}(g)\colon \mathcal{F}(B) \to \mathcal{F}(A)$:

$$\mathcal{F}(g)(s) = e_{\mathcal{F}}\langle s, g\rangle.$$

Clauses 2–3 express contravariant functoriality of $\mathcal{F}(g)$.

As a trivial example, used in Corollary 3.5, the constant singleton presheaf on $\mathbf{C}$ is the presheaf with $1_{C_0}\colon C_0 \to C_0$ as set of sections, and (up to isomorphism) presheaf action $d_1\colon C_1 \to C_0$.

A *natural transformation* of presheaves $\eta\colon \mathcal{F} \to \mathcal{G}$ is a function over C_0

$$\begin{array}{ccc} F_0 & \xrightarrow{\ \eta\ } & G_0 \\ & {\scriptstyle\gamma_0}\searrow \quad \swarrow{\scriptstyle\gamma_0'} & \\ & C_0 & \end{array} \qquad \gamma_0 = \gamma_0'\eta$$

which commutes with the actions $e_{\mathcal{F}}$ and $e_{\mathcal{G}}$ in the obvious way.

Informally, presheaves on $\mathbf{C}$ form a complete and cocomplete locally small category. But MacSet cannot formalize this claim. By itself, MacSet must make more cautious statements:

DEFINITION 3.3. *An I-*indexed set *of presheaves $\{\mathcal{F}_i | i \in I\}$ on a small category $\mathbf{C}$ is a $C_0 \times I$-indexed set $\gamma_0\colon F_0 \to C_0 \times I$ with an I-indexed action $e_{\mathcal{F}}\colon F_1 \to F_0$ where now*

$$F_1 = \{\langle s, f, i\rangle \in F_0 \times C_1 \times I \mid \gamma_0(s) = \langle d_1(f), i\rangle\}.$$

THEOREM 3.4. *Any I-indexed set of presheaves $\{\mathcal{F}_i | i \in I\}$ on a small category $\mathbf{C}$ has an $I \times I$-indexed set $\{\eta\colon \mathcal{F}_i \to \mathcal{F}_j | \langle i, j\rangle \in I\}$ of all natural transformations between them.*

Proof. For any indexed set of presheaves $\gamma_0\colon F_0 \to C_0 \times I$, the graph of each natural transformation between two of them is a subset of $F_0 \times F_0$. So the set of all graphs of transformations is a subset of the power set $\mathcal{P}(F_0 \times F_0)$. From here the proof is like either of our two proofs of Theorem 3.2, noting the natural transformations are defined by equations on the functions defining $\mathbf{C}$ and $\mathcal{F}$, thus with quantifiers bounded by (products of) the sets C_0, C_1, F_0, F_1. □

COROLLARY 3.5. *Every I-indexed set of presheaves $\{\mathcal{F}_i | i \in I\}$ on a small category $\mathbf{C}$ has an I-indexed set of limits $\{\varprojlim(\mathcal{F}_i) | i \in I\}$, and of colimits $\{\varinjlim(\mathcal{F}_i) | i \in I\}$.*

Proof. The indexed set of limits is just the indexed set of transformations from the constant singleton presheaf to $\{\mathcal{F}_i | i \in I\}$. The case of colimits is the indexed set of transformations in the other direction. (These are essentially the proofs in Mac Lane, 1998, pp. 110 and 112 ex. 8.) □

Given parallel natural transformations $\eta, \iota\colon \mathcal{F} \to \mathcal{G}$ of presheaves the usual constructions of a presheaf equalizer and a presheaf coequalizer work in MacSet (Mac Lane, 1998, p. 115). And every indexed set $\{\mathcal{F}_i | i \in I\}$ of presheaves on small category $\mathbf{C}$ has a coproduct presheaf $\coprod_i \mathcal{F}$ with set of sections given by projection to C_0:

$$(\textstyle\coprod_i \mathcal{F}_i)_0 = F_0 \xrightarrow{\ \coprod \gamma_0\ } C_0 \quad = \quad F_0 \xrightarrow{\ \gamma_0\ } C_0 \times I \xrightarrow{\ p_0\ } C_0 .$$

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For each A, the value $(\coprod_i \mathcal{F}_i)(A)$ is the disjoint union of the values $\mathcal{F}_i(A)$ for $i \in I$. So the action $e_{\mathcal{F}}\colon F_1 \to F_0$ is also the action for $\coprod \mathcal{F}$. The usual construction of a product of an indexed set of presheaves also works in MacSet, using the function set F_0^I to provide an ambient set.

The explicit constructions of $\prod_i \mathcal{F}_i$ and $\coprod_i \mathcal{F}_i$, plus the obvious canonical choice for equalizers and coequalizers, define a unique canonical limit presheaf and colimit presheaf for every diagram of presheaves.

3.3. The Yoneda lemma. Each object B of a small category $\mathbf{C}$ *represents* a presheaf R_B assigning to each object A of $\mathbf{C}$ the set

$$R_B(A) = \mathrm{Hom}_{\mathbf{C}}(A, B)$$

of all arrows from A to B. Each $\mathbf{C}$ arrow $f\colon A' \to A$ gives a function

$$R_B(f) : \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{C}}(A', B)$$

defined by $R_B(f)(g) = gf$. There is even a C_0-indexed family of all representable presheaves R_B, namely C_1 with the domain and codomain functions:

$$C_1 \xrightarrow{\langle d_0, d_1\rangle} C_0 \times C_0\ .$$

Any arrow $h\colon B \to D$ of $\mathbf{C}$ induces a natural transformation of presheaves in the same direction, defined in the natural way:

$$R_h\colon R_B \to R_D \qquad R_h(g) = hg \quad \text{for all } g \in R_B.$$

This operation is functorial in that $R_h R_k = R_{hk}$ and $R_{(1_B)} = 1_{(R_B)}$.

The simplest Yoneda lemma says for any presheaf $\mathcal{F}$ on $\mathbf{C}$ and object B of $\mathbf{C}$, natural transformations $R_B \to \mathcal{F}$ correspond naturally to elements of $\mathcal{F}(B)$. Mac Lane (1998, p. 59) has a proof suitable for MacSet. So the representables are *generators*: any two distinct natural transformations of presheaves $\eta \neq \theta\colon \mathcal{F} \to \mathcal{G}$ are distinguished by some natural transformation $\nu\colon R_B \to \mathcal{F}$.

$$R_B \xrightarrow{\ \nu\ } \mathcal{F} \underset{\theta}{\overset{\eta}{\rightrightarrows}} \mathcal{G} \qquad\qquad \eta\nu \neq \theta\nu.$$

A stronger Yoneda lemma says every presheaf is a colimit of presheaves R_B. The elementary proof by Johnstone (1977, p. 51) is easily formalized in MacSet. All of this generalizes to indexed sets of presheaves just like Theorems 3.2 and 3.4.

3.4. Sites. We take the notion of coverage from Johnstone (2002, C 2.1.1). Johnstone discusses the relation with Grothendieck's topologies and pretopologies.

DEFINITION 3.6.

(1) A coverage *$J \subseteq C_0 \times \mathcal{P}(C_1)$ on a small category $\mathbf{C}$ relates objects A to sets of arrows with codomain A. The sets related to A are called the J-covering sets for A and must meet this covering condition: For any arrow $g\colon A \to B$ in $\mathbf{C}$ and J-covering set S of B, there exists a J-covering set S' of A such that for every $\mathbf{C}$ arrow $h \in S'$ the composite gh in $\mathbf{C}$ factors through some $\mathbf{C}$ arrow $f \in S$.*

(2) A small site *$\langle \mathbf{C}, J\rangle$ is a small category and a coverage on it.*

The coverages of $\mathbf{C}$ form a subset of the power set $\mathcal{P}(C_0 \times \mathcal{P}(C_1))$ with defining condition explicitly bounded in this definition.

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A *J-sheaf* on a small site $\langle \mathbf{C}, J \rangle$ is a presheaf $\mathcal{F}$ meeting a compatibility condition: for every J-covering set $\{f_i : A_i \to A | i \in I\}$ the value $\mathcal{F}(A)$ is an equalizer

$$\mathcal{F}(A) \xrightarrow{\nu} \prod_i \mathcal{F}(A_i) \underset{\theta}{\overset{\eta}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(A_i \times_A A_j) .$$

The sheaves on a small site form a proper class.

The usual proofs work verbatim in MacSet to show each presheaf $\mathcal{F}$ on a small site $\langle \mathbf{C}, J \rangle$ has an *associated sheaf* $L\mathcal{F}$ and natural transformation $i : \mathcal{F} \to L\mathcal{F}$ such that every natural transformation $\eta : \mathcal{F} \to S$ to a J-sheaf S factors uniquely through i. This universal property shows each natural transformation of presheaves $\theta : \mathcal{F} \to \mathcal{G}$ induces a natural transformation of the J-sheaves $L\theta : L\mathcal{F} \to L\mathcal{G}$.

THEOREM 3.7. *All theorems of elementary topos theory hold for sheaves over any small site in MacSet. See for example (Johnstone, 1977).*

Proof. The elementary topos axioms and proofs involve only bounded constructions on objects and arrows. □

Routine attention to ambient sets in MacSet shows further:

LEMMA 3.8. *Every I-indexed set of presheaves $\{\mathcal{F}_i | i \in I\}$ over a site $\langle \mathbf{C}, J \rangle$ has an I-indexed set of associated sheaves $\{L\mathcal{F}_i | i \in I\}$.*

Proof. This is more or less obvious in that the desired indexed set $\{L\mathcal{F}_i | i \in I\}$ is bounded below some calculable rank n over the set $C_0 \cup F$, and is defined within $V_n(C_0 \cup F)$ by equations quantifying over C_0, C_1, F. But for more detail, following Mac Lane & Moerdijk (1992, p. 129), it suffices to show this for the operator $\mathcal{F} \mapsto \mathcal{F}^+$ on presheaves in place of $\mathcal{F} \mapsto L\mathcal{F}$, since $L\mathcal{F} = \mathcal{F}^{++}$.

For each object A of $\mathbf{C}$ the set of sections of $\mathcal{F}^+$ over A is a set of equivalence classes of compatible families of sections of $\mathcal{F}$. The relevant families are those that lie over covers of A. But the power set $\mathcal{P}(F_0)$ serves as one ambient set containing all the families we want, for all objects A. The equivalence classes all lie in the iterated power set $\mathcal{P}^2(\mathcal{F}_0)$. The quantifiers defining the set of sections $\gamma : (\mathcal{F}^+)_0 \to C_0$ as a subset of $\mathcal{P}^2(\mathcal{F}_1) \times C_0$ are bounded by C_0, C_1, and F_0. Analogous treatment works for the action on $\mathcal{F}^+$.

Now suppose given an indexed set of presheaves $\{\mathcal{F}_i | i \in I\}$, formally a $C_0 \times I$-indexed set $\gamma_0 : F_0 \to C_0 \times I$ with I-indexed action $e_{\mathcal{F}} : F_1 \to F_0$ on the set

$$F_1 = \{\langle s, f, i \rangle \in F_0 \times C_1 \times I \mid \gamma_0(s) = \langle d_1(f), i \rangle\}.$$

The set of sections of all the associated sheaves $\{L\mathcal{F}_i | i \in I\}$ is formed in the single iterated power set $\mathcal{P}^2(F_0)$ for this F_0 and the set of actions for all the associated sheaves is similarly compounded from C_0, C_1, F_0, F_1. □

3.5. Size of sites. While we have handled small sites, most textbooks and published proofs make number theoretic sites proper classes. The issue is not *gros* versus *petit* sites. Those have the same set theoretic size and only differ in the dimensions of fibers. The issue is that cutting proper class sites down to sets is not trivial. A scheme site local on the fibers is closed under set-sized disjoint unions, and so cannot be small. Often, *quasi-compactness* implies some small site has the same category of sheaves. See EGA I 6.3.1 or Tamme (1994, p. 90).

The comparison lemma, SGA 4 III.4.1, our Theorem 7.12, works for many cases. Verdier SGA 4 III.0 notes the use of this lemma "obliges us to certain contortions." Milne (2016, p. 57) puts it well: "In any specific situation such set-theoretic questions will cause no special difficulty, but to be both rigorous and general one should use universes." §7 shows that our universe in the class theory MacClass works more or less exactly as well (and as awkwardly) for the actual purposes of SGA as Grothendieck's notion of universes does.

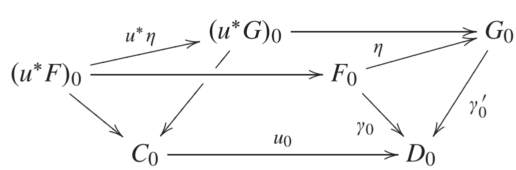
3.6. Functoriality of presheaves. Grothendieck and Verdier SGA 4 I.5, and Verdier SGA 4 III.1–3 prove various relations between small site functors, and presheaf functors. All are all provable in MacSet. Suitable bounds are obvious in SGA, and many are explicit in Johnstone (1977, Chap. 2).

The basic case is composing a presheaf $\mathcal{F}$ on $\mathbf{D}$ with a functor $u\colon \mathbf{C} \to \mathbf{D}$ to get a presheaf $u^*\mathcal{F}$ on $\mathbf{C}$. If you think of $\mathcal{F}$ as a set-valued functor $\mathcal{F}\colon \mathbf{D}^{op} \to \mathcal{S}et$ then the composite would be the composite of functors:

$$\mathbf{C}^{op} \xrightarrow{u^{op}} D^{op} \xrightarrow{\mathcal{F}} \mathcal{S}et.$$

But the Grothendieck construction formalizes $\mathcal{F}$ as a D_0-indexed set $\gamma_0\colon F_0 \to D_0$ plus an action $e_{\mathcal{F}}\colon F_1 \to F_0$ and so defines $u^*\mathcal{F}$ by the pullback of $\gamma_0\colon F_0 \to D_0$ along $f_0\colon C_0 \to D_0$. Details are in Mac Lane & Moerdijk, 1992, p. 243 and other references. So, if $\mathcal{F}$ has object part $F_0 = \{F_A \mid A \in \mathbf{D}_0\}$ then up to isomorphism $(u^*\mathcal{F})_0 = \{F_{u(B)} \mid B \in \mathbf{C}_0\}$. The action of the arrows C_1 on $u^*\mathcal{F}$ is the action of D_1 on $\mathcal{F}$ composed with the arrow part $U_1\colon C_1 \to D_1$ of u.

A natural transformation $\eta\colon \mathcal{F} \to \mathcal{G}$ of presheaves on $\mathbf{D}$ is a D_0 indexed set of functions. Its image $u^*\eta\colon u^*\mathcal{F} \to u^*G$ is η re-indexed over C_0 by pullback:



Here $(u^*\mathcal{F})_0 \to C_0$ and $(u^*\mathcal{G})_0 \to C_0$ are the pullbacks of γ_0 and γ_0' along u_0, and $u^*\eta$ is the unique arrow making the diagram commute.

Thus, for a fixed functor $u\colon \mathbf{C} \to \mathbf{D}$ the set theory MacSet proves u^* is a well-defined functor from presheaves on $\mathbf{D}$ to presheaves on $\mathbf{C}$. Lightly adapting either SGA 4 I.5 and 4 III.1–3, or the relevant parts of Johnstone (1977) proves in MacSet that u^* has well-defined left and right adjoint functors $u^!$ and u_* (albeit these functors and presheaf categories are proper classes, not sets). We spell out u_* for indexed sets of presheaves. Analogous results hold for u^* and $u^!$:

LEMMA 3.9. *For any functor of small categories $u\colon \mathbf{C} \to \mathbf{C}'$ the functor u_* right adjoint to u^* takes each indexed set $\{\mathcal{F}_i | i \in I\}$ of presheaves on $\mathbf{C}$ to an indexed set of presheaves $\{u_*(\mathcal{F}_i) | i \in I\}$ on $\mathbf{C}'$.*

Proof. For each object Y of $\mathbf{C}'$, Grothendieck and Verdier (SGA 4 I.5) form a small category $\mathcal{I}^u_Y$, now more often written $(u \downarrow Y)$ the comma category of u over Y. Objects of $(u \downarrow Y)$ are pairs $\langle X, m\rangle$ with $m\colon u(X) \to Y$, and arrows $\xi\colon \langle X, m\rangle \to \langle X', m'\rangle$ are defined by commutative triangles

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$$
\begin{array}{ccccc}
X & & u(X) & \xrightarrow{\quad m \quad} & \\
\Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle u(\xi)} & & Y. \\
x' & & u(X') & \xrightarrow[\quad m' \quad]{} &
\end{array}
$$

A projection functor $pr_Y \colon (u \downarrow Y) \to \mathbf{C}$ takes $\langle X, m\rangle$ to X.

Composing pr_Y with a presheaf $\mathcal{F}$ on $\mathbf{C}$ gives presheaf $pr_Y^*\mathcal{F}$. Define $(u_*\mathcal{F})(Y)$ as $\varprojlim pr_Y^*\mathcal{F}$ (Corollary 3.5). Every $f \colon Y \to Y'$ in $\mathbf{C}'$ induces a functor $(u \downarrow f)$ from $(u \downarrow Y)$ to $(u \downarrow Y')$, and so a function $(u_*\mathcal{F})(f) \colon (u_*\mathcal{F})(Y') \to (u_*\mathcal{F})(Y)$.

Each set $u_*\mathcal{F}(Y)$ can be taken as a subset of the function set $F_0^{C_0 \times C_1'}$. So the graph of the structure map $\gamma : (u_*\mathcal{F})_0 \to C_0'$ is a subset of $F_0^{C_0 \times C_1'} \times C_0'$. The quantifiers defining this subset are bounded by C_0, C_1, C_0', C_1', and F_0. Analogous treatment works for the action, so $u_*\mathcal{F}$ is a presheaf on $\mathbf{C}'$.

An I-indexed set $\{\mathcal{F}_i | i \in I\}$ is a function $\gamma_0 \colon F_0 \to C_0 \times I$ with I-indexed action. Each $u_*(\mathcal{F}_i)$ can be constructed this way, but all working in the single ambient set $F_0^{C_0 \times C_1'} \times C_0'$ for this set F_0, so as to define a single set $\{u_*(\mathcal{F}_i) | i \in I\}$. □

The Grothendieck construction of presheaves makes u^* pseudofunctorial in u. Given any $v \colon \mathbf{B} \to \mathbf{C}$, the composite functors v^*u^* and $(uv)^*$ are naturally isomorphic but generally not equal.

3.7. Étale fundamental groups. A topological space X has *covering spaces* as e.g., a helix covers a circle. Symmetries of a suitable cover of X form its (topological) fundamental group, like a Galois group, and reveal much about X. The *finite étale covers* of a scheme X give uncannily good analogues to topological covers, and the corresponding étale fundamental groups include Galois groups as special cases (Grothendieck, 1971).

The theory of finite étale covers is elementary algebra (EGA IV). The fundamental group uses a category of "all" finite étale covers of a scheme X, meaning all *up to isomorphism*. Since these covers are given by finitely generated extensions of coordinate rings on X, MacSet provides a sufficient category using the set of all extensions generated by finite subsets of one fixed countably infinite set G.

3.8. Injectives and cohomology groups. Baer (1940) used replacement to prove every module embeds in an injective module. Eckmann & Schopf (1953) proved it without replacement, but requiring choice to show divisible Abelian groups are injective (Blass, 1979). Grothendieck (1957a) adapted Baer's proof to sheaves of modules on topological spaces in a way that actually works in any Grothendieck topos. Because it uses choice the Eckmann-Schopf proof does not lift directly to the topological case let alone all Grothendieck toposes. Barr (1974) overcame this by showing every Grothendieck topos E has *Barr covers* satisfying choice.

A series of lemmas leading to Theorem 3.14 shows MacSet suffices to formalize that proof, for sheaves of modules over any site, not only for single injective embeddings but for infinite injective resolutions.

3.8.1. Resolutions in sets. Standard proofs work in MacSet to show every Abelian group embeds in a divisible one, and every divisible Abelian group is injective. That is all there is to know about injective resolution of Abelian groups, since quotients of divisible groups are divisible, so every embedding of an Abelian group A into a divisible I_0 gives a length one injective resolution:

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$$A \rightarrowtail I_0 \longrightarrow I_0/A \longrightarrow 0 \, .$$

Injective modules over a ring are more subtle, and can require infinite resolutions. We use a result first published in Artin (1962):

LEMMA 3.10. *If a functor* $\mathbf{F}\colon \mathbf{B} \to \mathbf{A}$ *has a left exact left adjoint* $\mathbf{G}\colon \mathbf{A} \to \mathbf{B}$ *with monic unit and each object in* $\mathbf{B}$ *embeds in an injective then so does each in* $\mathbf{A}$.

Proof. If units are monic, every monic $\mathbf{G}(A) \rightarrowtail B$ has monic adjunct $A \rightarrowtail \mathbf{F}(B)$. Since $\mathbf{G}$ preserves monics, $\mathbf{F}$ preserves injectives. If object A in $\mathbf{A}$ has a monic $\mathbf{G}(A) \rightarrowtail I$ to an injective in $\mathbf{B}$, the adjunct $A \rightarrowtail \mathbf{F}(I)$ is monic. □

COROLLARY 3.11. *For any ring R, every R-module embeds in an injective.*

Proof. Let $\mathbf{F}$ take each Abelian group A to the R-module $Hom_{\mathbb{Z}}(R, A)$ of additive functions from R to A, with $r \cdot f$ defined by $(r \cdot f)(x) = f(r \cdot x)$. It has left exact left adjoint $\mathbf{G}$ the underlying group functor. For each M the unit η_M takes each $m \in M$ to the function $r \mapsto r \cdot m$, so is monic. □

Lemma 3.10 and Corollary 3.11 produce injective resolutions of any finite length n for any module M. That is exact sequences

$$M \rightarrowtail I_0 \longrightarrow \cdots \longrightarrow I_n$$

with all I_i injective. Define sequences I_i and M_i inductively:

(1) Set $M_0 = M$.
(2) Embed M_i as an additive group into a divisible group $M_i \rightarrowtail M_{di}$.
(3) Form the injective R-module $I_i = Hom_{\mathbb{Z}}(R, M_{di})$ with monic $M_i \rightarrowtail I_i$.
(4) Start again, with the quotient $M_{i+1} = I_i/M_i$.

Textbooks immediately conclude there are infinite injective resolutions, by implicit use of (countable) replacement. MacSet proves the same conclusion, but only after bounding the infinite procedure inside one ambient set for each module M.

The ambient will be the function set $M^{\mathbb{Z}\times R^{\mathbb{N}}}$ which has an R-module structure induced by M. Here $R^{\mathbb{N}}$ is the set of infinite sequences in R. Say a function $f\colon \mathbb{Z} \times R^{\mathbb{N}} \to M$ is *cut off at* $n \in \mathbb{N}$ if $f(m, \sigma) = 0$ for every sequence σ which does *not* have $\sigma(i) = 0$ for all $i \geq n$. In effect a function cut off at n is an element of $M^{\mathbb{Z}\times R^n}$. So, a function cut off at $n+1$ can also be regarded as a function from R to the set $M^{\mathbb{Z}\times R^n}$ of functions cut off at n.

Also, notice Step 2 is idle for $i \geq 1$ since all I_i and all $M_{i+1} = I_{i+1}/I_i$ are divisible groups. So it suffices to give an infinite injective resolution for each module M with divisible underlying group. For this case $M_i = M_{di}$ for all $i \in \mathbb{N}$.

For any ring R, and R-module M with divisible underlying group, define this induction parallel to the one above:

(1′) Let the subset $N_0 \subset M^{R^{\mathbb{N}}}$ contain just the additive functions cut off at 0. In effect these are additive functions $\mathbb{Z} \to M$, so $N_0 \cong M$.
(1″) Define equivalence relation E_0 as the identity on N_0. The point is

$$M \cong N_0 \cong N_0/E_0.$$

(3′) Given the subset $N_i \subset M^{R^{\mathbb{N}}}$ with every function cut off at i, and equivalence relation E_i on it, define a certain subset $J_i \subset M^{R^{\mathbb{N}}}$ of functions which are cut

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off at $i+1$. Namely, think of these as functions $R \to M^{\mathbb{Z}\times R^n}$. Let Ji contain just those whose values all lie in N_i and which are additive when seen as functions $R \to N_i/E_i$. Let Q_I be the pointwise equivalence relation making functions $R \to N_i$ equivalent iff they are equal as functions $R \to N_i/E_i$.

(3″) There is a natural monic $h\colon N_i \rightarrowtail J_i$ where for each $g \in N_i$ the value $h(g)$ is the unique R-linear function $R \to N_i/E_i$ taking $1 \in R$ to g.

(4′) Define $N_{i+1} = J_i$ with E_{i+1} the smallest equivalence relation containing both Q_i and the relation induced by the submodule $h\colon N_i \rightarrowtail J_i$.

For every $i \in \mathbb{N}$ the quotient N_i/E_i is isomorphic as R-module to the module M_i above, while each J_i/Q_i is isomorphic to I_i above, So this gives an isomorphic copy of the resolution by I_i above. Bounded separation suffices to show this infinite resolution is one set, since $M^{\mathbb{Z}\times R^{\mathbb{N}}}$ suffices as ambient set, and quantifier bounds are explicit in the steps of the induction.

3.8.2. Resolutions over sites. Now let $\langle \mathbf{C}, J\rangle$ be any site, and $\mathcal{R}$ any sheaf of rings on it. We want to show sheaves of modules on $\mathcal{R}$ have infinite injective resolutions. The argument of §3.8.1 works in any elementary topos with natural numbers and choice, so it works for sheaves over any site whose sheaves satisfy choice in the obvious way: every sheaf epimorphism has a right inverse.[5] So it works over any Barr covering site of $\langle \mathbf{C}, J\rangle$. Compare van Osdol (1975). We must show in MacSet every site has a Barr covering site and each infinite resolution descends (as a single set) along that Barr cover. The first is clear from the construction by Mac Lane & Moerdijk (1992, pp. 511–513).

COROLLARY 3.12. *For any surjection of ringed toposes $f^* \dashv f_*\colon (\mathcal{B}, \mathcal{R}') \to (\mathcal{A}, \mathcal{R})$ if every $\mathcal{R}'$ module embeds in an injective then so does every $\mathcal{R}$ module, and f_* preserves injectives.*

Proof. Lemma 3.10, noting topos surjections have monic unit. □

LEMMA 3.13. *For any geometric morphism $f^* \dashv f_*\colon \mathcal{B} \to \mathcal{A}$ where $\mathcal{B}$ satisfies the axiom of choice, f_* preserves all exact sequences of modules over any ring.*

Proof. Direct image functors preserve monics. In the choice topos $\mathcal{B}$ every module quotient $q\colon \mathcal{M} \twoheadrightarrow \mathcal{M}/\mathcal{J}$ has a right inverse function $\mathcal{M}/\mathcal{J} \to \mathcal{M}$ (generally not a homomorphism), so $f_*(q)$ also does, so $f_*(q)$ is an epimorphism and thus a quotient. □

THEOREM 3.14. *For any sheaf of rings $\mathcal{R}$ on any site $\langle \mathbf{C}, J\rangle$, every sheaf of $\mathcal{R}$-modules $\mathcal{M}$ has an infinite injective sheaf resolution.*

Proof. Over any Barr cover of $\langle \mathbf{C}, J\rangle$, $f^*(\mathcal{M})$ has an infinite injective resolution existing as a single set

$$f^*(\mathcal{M}) \rightarrowtail \mathcal{I}_0 \longrightarrow \cdots \longrightarrow \mathcal{I}_n \longrightarrow \cdots$$

By Lemma 3.13 its f_* image is exact and since the unit $\mathcal{M} \rightarrowtail f_*f^*(\mathcal{M})$ is monic this is an injective resolution:

$$\mathcal{M} \rightarrowtail f_*(\mathcal{I}_0) \longrightarrow \cdots \longrightarrow f_*(\mathcal{I}_n) \longrightarrow \cdots$$

By Lemmas 3.8 and 3.9 this resolution exists as a single set. □

[5] This requires only global choice, not the internal axiom of choice (Johnstone, 1977, p. 261).

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3.8.3. Cohomology groups. So MacSet proves every sheaf of modules has an infinite injective resolution, existing as one set. Indeed it can define a specific resolution for any module over a given site. The axiom of choice in MacSet is used to verify the construction works, specifically by showing divisible groups and Barr covers have the requisite properties, but choice is not used to specify the resolution. The usual formalities of homological algebra show cohomology groups are functorial, exact, and effaceable. So MacSet can specify a long exact cohomology sequence for each short exact sequence of sheaves of modules. Standard results on Čech cohomology and spectral sequences also follow.

By the same method as Lemmas 3.8 and 3.9, indexed sets of sheaves of R-modules have indexed sets of infinite injective resolutions.

§4. Classes of sets and collections of classes. The set theory MacSet can talk about large categories and other proper classes in the same way ZFC can. In both theories: "proper classes do not exist, and expressions involving them must be thought of as abbreviations for expressions not involving them" (Kunen, 1983, p. 24). This limits how much either theory can prove about proper classes. Neither MacSet nor ZFC can quantify over proper classes or form collections of proper classes.

As the central example, MacSet proves existence of many small sites $\langle \mathbf{C}, J \rangle$, and many sheaves and sheaf maps. But no actual category of all sheaves and sheaf maps over $\langle \mathbf{C}, J \rangle$ can exist in MacSet or ZFC. The sheaves over $\langle \mathbf{C}, J \rangle$ provably form a proper class. In other words no Grothendieck topos actually exists in MacSet or ZFC. Of course MacSet can quantify over small sites, and so get some effects of quantifying over toposes. But MacSet cannot state theorems of SGA which actually quantify over toposes or form larger categories with toposes as objects. So, we take sets as one type, and add classes of sets as the next higher type, and collections of classes as the next higher. This idea, using devices from Takeuti (1978, 1987), gives *Mac Lane Class Theory* (MacClass). It provides all of the large structure tools of SGA, yet is conservative over MacSet.

4.1. Mac Lane Class Theory. Mac Lane Class Theory (MacClass) is based on MacSet. Similar class theories can be based on ETCS, or Finite-Order Arithmetic. They can be fully formalized in the Simple Type Theory (STT) of Takeuti, *Proof Theory* §20 (1987).

There is a hierarchy of types:

- There is a ground type *Set*.
- For every type τ there is a type $[\tau]$ (for "sets of" things of type τ though of course not sets in the sense of *Set*).

In particular, [*Set*] is the type of classes of sets. And [[*Set*]] could be called the type of classes of classes, or better the type of *collections*:

$$Class = [Set] \text{ and } Collection = [Class].$$

A simultaneous induction defines terms and formulas. As to terms:

- Terms of MacSet are terms of type *Set* of MacClass. (We use constants $\emptyset$, $\mathbb{N}$ and function symbols $\cup$, $\times$, $\mathcal{P}$, and could use bounded set abstracts, though all these can be eliminated if the reader prefers.)
- Variables of any type are terms of that type.

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- Let $\Psi(v_1)$ be any formula with free variables including the exhibited v_1 of type τ_1, and with no quantifiers except (possibly) over *Set* variables. Then $\{v_1|\Psi(v_1)\}$ is a term of type $[\tau_1]$, called a *set theoretic abstract*.

As to formulas:

- Formulas of MacSet are formulas of MacClass, and: for terms t_1, t_2 of type τ, and t_3 of type $[\tau]$ there are formulas $t_1 = t_2$ and $t_1 \in t_3$.
- If A and B are formulas, then $(\neg A)$, $(A \& B)$, $(A \vee B)$, $(A \supset B)$, $\forall x A(x)$, and $\exists x A(x)$ are formulas.

For any formula $\Psi(v_1)$ and term t_1 with the same type as v_1, Simple Type Theory stipulates that $t_1 \in \{v_1|\Psi(v_1)\}$ and $\Psi(t_1)$ imply each other (Takeuti, 1987, § 20). This rule, plus the standard rules of $\exists$, makes the following comprehension statement provable for every set theoretic formula $\Psi(v_1)$ (which may have free variables besides the exhibited v_1):

$$\exists \alpha \left(v_1 \in \alpha \ \leftrightarrow \ \Psi(v_1) \right).$$

Variables of type *Set* will be upper or lower case italics x, A, just as in MacSet. Variables of type *Class* will often be calligraphic $\mathcal{A}, \mathcal{B} \ldots$; and variables of type *Collection* often fraktur $\mathfrak{A}, \mathfrak{B} \ldots$. Membership signs may be subscripted to emphasize typing. So $A \in_0 B$ says set A is in set B, while $A \in_1 \mathcal{A}$ says set A is in class $\mathcal{A}$. And $\mathcal{A} \in_2 \mathfrak{B}$ says class $\mathcal{A}$ is in collection $\mathfrak{B}$.

For example these formulas define subclass inclusion and subcollection inclusion:

$$\mathcal{A} \subseteq_1 \mathcal{B} \leftrightarrow \forall x \, (x \in_1 \mathcal{A} \rightarrow x \in_1 \mathcal{B}) \tag{5}$$

$$\mathfrak{A} \subseteq_2 \mathfrak{B} \leftrightarrow \forall \mathcal{X} \, (\mathcal{X} \in_2 \mathfrak{A} \rightarrow \mathcal{X} \in_2 \mathfrak{B}). \tag{6}$$

In discussing locally small categories we often refer to subsets of a class:

$$A \subseteq_{01} \mathcal{B} \leftrightarrow \forall x \, (x \in_0 A \rightarrow x \in_1 \mathcal{B}). \tag{7}$$

A formula is *set theoretic* if it quantifies only over sets, while it may include terms of any type. Formulas (5) and (7) are set theoretic and Formula (6) is not. Subcollection inclusion $\subseteq_2$ is well defined but not set theoretic.

Since MacClass uses only set theoretic formulas in abstracts, Gentzen cut elimination shows MacClass is conservative over MacSet just as Gödel-Bernays set theory (GB) is conservative over ZFC.[6] MacClass and GB both quantify over classes in proofs. They just do not quantify over classes in definitions of sets or classes.

4.1.1. Axioms and proofs in MacClass. The axioms of MacClass are just the axioms of MacSet. Proofs in MacClass use those plus the inference rules of Simple Type Theory as sketched above. Those rules are basically:

(1) The formulas $t_1 \in \{v_1|\Psi(v_1)\}$ and $\Psi(t_1)$ imply each other, plus
(2) standard natural deduction rules for the logical connectives.

Details are in Takeuti (1987).

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[6] Takeuti (1978, p. 77f.) and (1987, p. 176). We do not need the far more complicated cut elimination theorem for full Simple Type Theory.

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§5. General category theory in MacClass. To use locally small categories we must compare classes and sets.

5.1. Sets, classes, and collections in MacClass.

(1) Every set A defines a class $\mathcal{A}$ with exactly the same elements:

$$\forall x(x \in_1 \mathcal{A} \leftrightarrow x \in_0 A).$$

We say informally the class $\mathcal{A}$ *is a set*.

(2) Ordered pairs of sets also define cartesian product for classes:

$$\mathcal{A} \times \mathcal{B} = \{\langle x, y\rangle \mid x \in_1 \mathcal{A} \,\&\, y \in_1 \mathcal{B}\}.$$

Thus $S \subseteq_{01} \mathcal{A}$ and $T \subseteq_{01} \mathcal{B}$ implies $S \times T \subseteq_{01} \mathcal{A} \times \mathcal{B}$.

(3) Using the sets 0,1, an ordered pair of classes can be coded as a class:

$$\langle \mathcal{A}, \mathcal{B}\rangle \;=\; (\mathcal{A} \times \{0\}) \cup (\mathcal{B} \times \{1\}).$$

(4) Every class $\mathcal{A}$ has a *power class* $\mathcal{P}_1(\mathcal{A})$ of all its subsets, and a *power collection* $\mathcal{P}_2(\mathcal{A})$ of all its subclasses:

$$\mathcal{P}_1(\mathcal{A}) = \{S | S \subseteq_{01} \mathcal{A}\} \quad \text{and} \quad \mathcal{P}_2(\mathcal{A}) = \{\mathcal{S} | \mathcal{S} \subseteq_1 \mathcal{A}\}.$$

(5) There is a collection $\mathcal{B}^{\mathcal{A}}$ of all functions $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ from class $\mathcal{A}$ to class $\mathcal{B}$.

A class function $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ is a triple $\langle \mathcal{A}, \mathcal{B}, \mathcal{R}\rangle$ with classes $\mathcal{A}, \mathcal{B}$, and subclass $\mathcal{R} \subseteq_1 \mathcal{A} \times \mathcal{B}$ functional from $\mathcal{A}$ to $\mathcal{B}$. The collection $\mathcal{B}^{\mathcal{A}}$ is defined by

$$\mathcal{B}^{\mathcal{A}} = \{\langle \mathcal{A}, \mathcal{B}, \mathcal{R}\rangle \mid \mathcal{R} \subseteq_1 \mathcal{A} \times \mathcal{B} \,\&\, (\forall x \in_1 \mathcal{A})(\exists! y \in_1 \mathcal{B})\, \langle x, y\rangle \in_1 \mathcal{R}\}.$$

All these definitions quantify only over sets. Clauses 2 and 3 extend to n-tuples.

The graph of $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ may be defined by some expression $\psi(x, y)$ relating each $x \in \mathcal{A}$ to a unique $y \in \mathcal{B}$. For MacClass to prove this $\mathcal{F}$ exists, $\psi(x, y)$ must quantify only over sets—but the set quantifiers need not be bounded.

DEFINITION 5.1. *For any set A and class $\mathcal{C}$, a* small $\mathcal{C}$-valued function *on A is any set which is the graph of some small function to a subset $H \subseteq_{01} \mathcal{C}$.*

Intuitively, a small $\mathcal{C}$-valued function is a small function to a subset of $\mathcal{C}$:

$$A \xrightarrow{\;f\;} Im(f) \subseteq_{01} \mathcal{C}.$$

Technically, though, we want small class-valued functions to be sets, so as to prove Theorem 5.2. So we define them as just functional graphs, and do not include the codomain $\mathcal{C}$ in encoding the function.

For contrast, the paradigm nonsmall function is the class function f from $\mathbb{N}$ to $\mathbb{N}$ with $f(n) = n$ when $\mathcal{P}^n(\mathbb{N})$ exists, and $f(n) = 0$ otherwise. Though the domain and codomain are both sets, this f is not small since its graph is a well-defined subclass of $\mathbb{N} \times \mathbb{N}$ without being a subset.

The discussion of completeness for locally small categories will use this result:

THEOREM 5.2. *For any set A and class $\mathcal{C}$, there is a class $\mathcal{C}^A$ of all small $\mathcal{C}$-valued functions on A; and a class $\mathcal{C}^{(\mathcal{S}et)}$ of all small $\mathcal{C}$-valued functions whatsoever.*

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Proof. Given A and $\mathcal{C}$, those classes are defined by set theoretic formulas:

$$\exists H\,(H \subseteq_{01} \mathcal{C}\ \&\ x \text{ is the graph of some small } A \to H)$$
$$\exists \text{ set } S\, \exists H\,(H \subseteq_{01} \mathcal{C}\ \&\ x \text{ is the graph of some small } S \to H). \qquad \square$$

5.2. The class category $\mathcal{C}at$ ***of all small categories.*** Many different familiar MacSet formulas define when a 6-tuple of sets $\langle C_0, C_1, d_0, d_1, id, m\rangle$ forms a small category $\mathcal{C}$. Write any one of them as

$$Cat(C_0, C_1, d_0, d_1, id, m).$$

A longer MacSet formula defines when $f_0 \colon C_0 \to C'_0$ and $f_0 \colon C_0 \to C'_0$ form a small functor $f \colon \mathbf{C} \to \mathbf{C}'$ between categories

$$Functor(f_0, f_1 : C_0, C_1, d_0, d_1, m; C'_0, C'_1, d'_0, d'_1, id', m').$$

So MacClass has a set-theoretic abstract for the class $\mathcal{C}at_0$ of all small categories:

$$\mathcal{C}at_0 \;=\; \{\langle C_0, C_1, d_0, d_1, id, m\rangle \mid Cat(C_0, C_1, d_0, d_1, id, m)\}.$$

A similar abstract gives the class $\mathcal{C}at_1$ of all small functors between small categories.

Then the obvious class functions $d_0, d_1 \colon \mathcal{C}at_1 \to \mathcal{C}at_0$, and partial function $m \colon \mathcal{C}at_1 \times \mathcal{C}at_1 \to \mathcal{C}at_1$ give a 6-tuple of classes describing the class category $\mathcal{C}at$ of all small categories:

$$\mathcal{C}at \;=\; \langle \mathcal{C}at_0, \mathcal{C}at_1, d_0, d_1, id, m\rangle.$$

5.3. The collection category $\mathfrak{Cat}$ ***of all class categories.*** A class category $\mathscr{C}$ is formally a 6-tuple of classes, also meeting a set theoretic condition.

$$\mathscr{C} \;=\; \langle \mathcal{C}_0, \mathcal{C}_1, d_0, d_1, id, m\rangle \quad \text{such that} \quad ClassCat(\mathcal{C}_0, \mathcal{C}_1, d_0, d_1, id, m).$$

This condition is set-theoretic since it only quantifies over members of $\mathcal{C}_0$ and $\mathcal{C}_1$. So, proceeding the same way as with the class category of all small categories, MacClass proves there is a collection category $\mathfrak{Cat}$ of all class categories:

$$\mathfrak{Cat} \;=\; \langle \mathfrak{Cat}_0, \mathfrak{Cat}_1, \mathfrak{d}_0, \mathfrak{d}_1, \mathfrak{id}, \mathfrak{m}\rangle.$$

Prima facie, the category axioms applied to entities of type higher than classes would require quantifying over entities of type higher than sets. I have not explored whether MacClass can say anything about higher type categories.

§6. The universe $\mathcal{U}$.

Take the class $\mathcal{U}$ of all sets as *universe*. We define locally small category and $\mathcal{U}$-category to be synonyms, where:

DEFINITION 6.1. *A class category* $\mathcal{C}$ *is* locally small *iff every set of objects* $S \subseteq_{01} \mathcal{C}_0$ *is the set of objects of a small full subcategory of* $\mathcal{C}$.

Thus the class of all set functions between sets provides a locally small category $\mathcal{S}et$. And $\mathcal{C}at$, described above, is locally small. But to work with this idea in detail, it is handy to define locally small functions.

DEFINITION 6.2. *A function* $f \colon \mathcal{A} \to \mathcal{B}$ *between classes* $\mathcal{A}$ *and* $\mathcal{B}$ *is* locally small *if, for every subset* $S \subseteq_{01} \mathcal{A}$*, restricting* f *to* S *gives a small* $\mathcal{B}$*-valued function*

$$f|_S \colon S \to \mathrm{Im}(f|_S) \subseteq_{0,1} \mathcal{B}.$$

LEMMA 6.3. *Immediately: if a locally small function has an inverse then the inverse is locally small; the composite of locally small functions is locally small; and the cartesian product of locally small functions is locally small.*

THEOREM 6.4. *A class category* $\langle C_0, C_1, d_0, d_1, id, m\rangle$ *is locally small iff every set of objects has a set of all arrows between them and the functions* d_0, d_1, m *are locally small.*

Proof. Necessity is immediate. For sufficiency we must show the stated conditions on C_0, C_1, d_0, d_1, m imply that id is also locally small. So let $A \subseteq_{01} C_1$ be the set of all arrows between some set of objects $S \subseteq_{01} C_0$. Then the set of identity arrows on objects in S is defined by the bounded condition

$$\{f \in_0 A \mid \forall g \in_0 A\big((d_0 g = d_1 f) \to m(g,f) = g\big)\}.$$

The graph of id restricted to S is similarly defined as a subset of $S \times A$. □

We define locally small functor and $\mathcal{U}$-functor to be synonyms, where:

DEFINITION 6.5. *A* $\mathcal{U}$-functor *is a functor* $f\colon \mathcal{C} \to \mathcal{D}$ *between* $\mathcal{U}$*-categories such that, for every small subcategory* $\mathbf{C}'$ *of* $\mathcal{C}$*, the restriction* $f|_{C'}$ *factors as a small functor to a small category* $\mathbf{D}'$ *followed by an inclusion:*

$$\mathbf{C}' \xrightarrow{\ f\ } \mathbf{D}' \rightarrowtail \mathcal{D}.$$

THEOREM 6.6. *A functor* $f\colon \mathcal{C} \to \mathcal{D}$ *between* $\mathcal{U}$*-categories is a* $\mathcal{U}$*-functor iff its arrow part* f_1 *is a locally small function.*

Proof. Necessity is immediate. For sufficiency note the object part f_0 of a functor $f\colon \mathcal{C} \to \mathcal{D}$ is the composite of the identity function id_C on $\mathcal{C}$ with f_1 and then the domain function d_{0D} on $\mathcal{D}$. Use Lemma 6.3. □

DEFINITION 6.7. *A* $\mathcal{U}$-adjunction *is a adjoint pair* $f \dashv g\colon \mathcal{A} \to \mathcal{B}$ *of* $\mathcal{U}$*-functors with locally small unit function* $\eta\colon \mathcal{A}_0 \to \mathcal{A}_1$.

THEOREM 6.8. *In a* $\mathcal{U}$-adjunction*, both sides of the bijection between* $\mathcal{A}$ *arrows* $fx \to y$ *and* $\mathcal{B}$ *arrows* $x \to gy$ *are locally small, as is the counit function* $\varepsilon\colon \mathcal{B}_0 \to \mathcal{B}_1$.

Proof. The functions in question are composites of locally small functions. □

THEOREM 6.9. *In any* $\mathcal{U}$*-adjunction* $f \dashv g\colon \mathcal{A} \to \mathcal{B}$ *the functor* g *preserves all small limits that exist in* $\mathcal{A}$*; and* f *preserves all small colimits that exist in* $\mathcal{B}$.

Proof. The usual argument, noting local smallness guarantees every class of objects or arrows that needs to be a set is a set. Specifically, any small diagram in $\mathcal{A}$ has small image under the composite functor $fg\colon \mathcal{A} \to \mathcal{A}$, and has a set of composites with the relevant counits. And analogously for small diagrams in $\mathcal{B}$. □

DEFINITION 6.10.

(1) A $\mathcal{U}$-equivalence *is an adjoint equivalence given by a* $\mathcal{U}$*-adjunction.*

(2) A $\mathcal{U}$-reflective subcategory *is a reflective subcategory which is given by a* $\mathcal{U}$*-adjunction.*

6.1. Sheaf and presheaf toposes. §3.2 defined presheaves on a small category $\mathbf{C}$ by the Grothendieck construction in MacSet. Now MacClass provides a context for presheaves as $\mathcal{S}et$-valued functors:

DEFINITION 6.11. *A* functor-presheaf *on a small category* $\mathbf{C}$ *is a contravariant* $\mathcal{U}$*-functor* $\mathcal{F}\colon \mathbf{C}^{op} \to \mathcal{S}et$; *and* $\mathcal{U}$*-natural transformations of them are given by set functions between the sets of values. Let* Presh$(\mathcal{C})$ *be the category of functor-presheaves on* $\mathbf{C}$ *and* $\mathcal{U}$*-natural transformations between them.*

Define $\widehat{\mathbf{C}}$ as the category of presheaves on $\mathbf{C}$ defined by the Grothendieck construction in §3.2. There is little need to distinguish Presh$(\mathcal{C})$ from $\widehat{\mathbf{C}}$:

THEOREM 6.12. *The category* Presh$(\mathbf{C})$ *is* $\mathcal{U}$*-equivalent to* $\widehat{\mathbf{C}}$ *by an explicitly defined adjoint pair of functors.* (*And this is natural in* $\mathbf{C}$ *though we do not linger to formalize that naturality.*)

Proof. Theorem 3.4 shows $\widehat{\mathbf{C}}$ is a $\mathcal{U}$-category; and a light modification shows the same for Presh$(\mathcal{C})$. Call a functor-presheaf $\mathcal{F}\colon \mathbf{C}^{op} \to \mathcal{S}et$ *disjoint* if all distinct $\mathbf{C}$ objects $a \neq b$ have disjoint values $\mathcal{F}(a)$ and $\mathcal{F}(b)$. Then $\widehat{\mathbf{C}}$ has a $\mathcal{U}$-functorial bijection to the full subcategory of disjoint functor-presheaves. Lemma 2.1 explicitly defines a $\mathcal{U}$-equivalence between that and $\widehat{\mathbf{C}}$. □

MacClass (unlike MacSet) makes presheaf categories genuine entities, namely classes. So now the work of §3.3 shows each small category has a *Yoneda embedding* $R\_\colon \mathbf{C} \to \widehat{\mathcal{C}}$, which takes each object B of $\mathbf{C}$ to the presheaf $R_B(\_)$ represented by B. MacClass proves many standard results on the Yoneda embedding which we will not explore.

THEOREM 6.13. *For every small category* $\mathbf{C}$, $\widehat{\mathbf{C}}$ *has limits and colimits for all small diagrams.*

Proof. Corollary 3.5 showed $\mathcal{S}et$ has all small limits and colimits. The usual reasoning works in MacSet (and thus in MacClass) to show limits and colimits in $\widehat{\mathbf{C}}$ are computed pointwise (Mac Lane 1998, p. 115). □

COROLLARY 6.14. *We can actually specify one particular 'chosen' limit and colimit for each small diagram in* $\widehat{\mathbf{C}}$.

Proof. The comment after Corollary 3.5 specifies one particular equalizer for every parallel pair of functions in the category of sets, and one particular product $\prod_i \mathcal{F}_i$ for each indexed set of sets. In the usual way this lets us choose one particular limit for each small diagram in the category of sets. $\widehat{\mathbf{C}}$. Since limits and colimits in $\widehat{\mathbf{C}}$ are computed pointwise, we get chosen ones by taking the chosen set-limit at each point. Analogous reasoning works for colimits. □

To be clear the chosen limits and colimits have no distinguishing categorical properties. But they are uniquely defined.

THEOREM 6.15. *The presheaf category* $\widehat{\mathbf{C}}$ *on any small site* $\langle \mathbf{C}, J\rangle$ *has a* $\mathcal{U}$*-reflective full subcategory of sheaves* $i\colon \widetilde{\mathbf{C}}_J \to \widehat{\mathbf{C}}$ *with left adjoint* $L\colon \widehat{\mathbf{C}} \to \widetilde{\mathbf{C}}_J$ *the associated sheaf functor. Further, L is left exact.*

Proof. Lemma 3.8 showed the associated sheaf functor $L\colon \widehat{\mathbf{C}} \to \widetilde{\mathbf{C}}_J$ is left $\mathcal{U}$-adjoint to the inclusion $\widetilde{\mathbf{C}}_J \rightarrowtail \widehat{\mathbf{C}}$. MacClass adds the existence of those $\mathcal{U}$-categories. Proofs in

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SGA 4 II or Mac Lane & Moerdijk (1992, p. 227ff.) easily adapts to show sheafification preserves finite limits. □

COROLLARY 6.16. *There is a limit and a colimit for every small diagram in* $\widetilde{\mathbf{C}}_J$. *Further, we can specify one particular 'chosen' limit and colimit for each small diagram in* $\widetilde{\mathbf{C}}_J$.

Proof. The usual sheer formalities show any limit in $\widehat{\mathbf{C}}$ of any diagram of sheaves is a sheaf, so it is a limit in $\widetilde{\mathbf{C}}_J$. And the sheafification of the colimit in $\widehat{\mathbf{C}}$ of any diagram of sheaves is a colimit in $\widetilde{\mathbf{C}}_J$. For the further claim, note the associated sheaf construction by Mac Lane & Moerdijk (1992, p. 129) is set-theoretically uniquely specified—even though the property of interest is isomorphism invariant. □

DEFINITION 6.17. *A* sheaf topos *is a category* $\widetilde{\mathbf{C}}_J$ *for a small site* $\langle \mathbf{C}, J \rangle$.

THEOREM 6.18. *Every sheaf topos is an elementary topos.*

Proof. This restates Theorem 3.7. □

THEOREM 6.19. *Every sheaf topos has a set G of generators, or in other words a small full subcategory* $\mathbf{G}$ *of generators.*

Proof. For every sheaf topos $\widetilde{\mathbf{C}}_J$, G can be the set of J-sheafifications LR_A of representable presheaves on objects A of $\mathbf{C}$. In other words $\mathbf{G}$ can be the image of $\mathbf{C}$ by the Yoneda functor $LR\_ : \mathbf{C} \to \widetilde{\mathbf{C}}_J$ to sheaves. □

A technical point leads towards the Giraud Theorem in §7:

DEFINITION 6.20. *In any category a set of arrows* $\{f_i : A_i \to A\}$ *is* jointly epimorphic *iff whenever* $h, k : A \to B$ *have* $hf_i = kf_i$ *for all* $i \in I$ *then* $h{=}k$.

THEOREM 6.21. *For any set A of arrows in a sheaf topos there is a set of all jointly epimorphic subsets of A.*

Proof. The standard proofs work in MacSet to show a set $\{f_i : F_i \to F | i \in I\}$ is jointly epimorphic in $\widetilde{\mathbf{C}}_J$ iff it is J-*locally onto* in this sense: For every object C of $\mathbf{C}$ and every section $s \in F(C)$ there is some J-covering set $\{C_k \to C | k \in K\}$ such that for every $k \in K$ the restriction of s to a section $s_k \in F(C_k)$ is in the image of some sheaf map $f_i(C_k) : F_i(C_k) \to F(C_k)$. This characterization is a bounded MacSet formula, so it defines a subset of the power set of A. □

For the Giraud Theorem, A will be the set of arrows of a small subcategory $\mathbf{G}$ of the sheaf topos.

6.2. Cohomology in MacClass. A sheaf of modules over a sheaf of rings on any small site $\langle \mathbf{C}, J \rangle$ is just a module $\mathcal{M}$ on a ring $\mathcal{R}$ in the sheaf topos $\widetilde{\mathbf{C}}_J$. All commutative algebra that does not use excluded middle or the axiom of choice holds in every Grothendieck topos by either Theorem 3.7 or 6.18.

For any ring $\mathcal{R}$ in any sheaf topos $\widetilde{\mathbf{C}}_J$, MacClass gives a $\mathcal{U}$-category $\mathcal{MOD}_{\mathcal{R}}$ of all $\mathcal{R}$-modules. The usual constructions of biproducts, kernels, and cokernels are bounded so they show in MacClass $\mathcal{MOD}_{\mathcal{R}}$ is an Abelian category.

§3.8 constructed cohomology groups $H^n(E, M)$ in MacSet. In MacClass these give cohomology functors $H^n : \mathcal{MOD}_{\mathcal{R}} \to \mathcal{AB}$ from the category of sheaves of modules to the category of ordinary Abelian groups.

MacClass can give the usual definition of a universal δ-functor (Hartshorne, 1997, p. 204). Every left exact functor $F\colon \mathcal{MOD}_{\mathcal{R}} \to \mathcal{AB}$ has *right derived functors*

$$F \cong R^0F,\ R^1F,\ \ldots,\ R^nF,\ \ldots$$

defined up to isomorphism either as a universal δ-functor over F, or as an effaceable δ-functor over F. See (Grothendieck, 1957a, p. 141).

The cohomology functors H^i are derived functors of the global section functor $\Gamma\colon \mathcal{MOD}_{\mathcal{R}} \to \mathcal{AB}$ which takes each module to its group of global sections.

6.3. Categories with selected limits. Corollary 6.16 shows every sheaf topos has limits and colimits for all small diagrams, but then further explicitly defines one selected limit and one selected colimit for each. Other such definitions are in fact available for various other specific $\mathcal{U}$-categories which we will not try to survey in any systematic way. And of course a given $\mathcal{U}$-category might have no limit at all, or no colimit, for some diagram. We will use MacClass to formalize the idea of selected small limits (or colimits) in an arbitrary $\mathcal{U}$-category.

DEFINITION 6.22. *For any $\mathcal{U}$-category $\mathcal{C}$ and small category* $\mathbf{D}$*, a* small diagram of shape $\mathbf{D}$ *in $\mathcal{C}$ is any small $\mathcal{C}$-valued functor $D\colon \mathbf{D} \to \mathcal{C}$.*

THEOREM 6.23. *For any $\mathcal{U}$-category $\mathcal{C}$ and small category* $\mathbf{D}$*, there is a class of all small diagrams of shape* $\mathbf{D}$ *in $\mathcal{C}$, and a class of all small diagrams in $\mathcal{C}$.*

Proof. Simple construction using Theorem 5.2. The conditions defining a small functor only quantify over sets. □

MacClass proves the analogues for all finite diagrams, all filtered diagrams, etc.

We take the definition of a cone over (or co-cone under) a small diagram in $\mathcal{C}$ as obvious. And remark that, for $\mathcal{U}$-categories $\mathcal{C}$, the definition of a limit cone (or co-limit co-cone) in a $\mathcal{C}$ only quantifies over sets. So:

THEOREM 6.24. *For any $\mathcal{U}$-category $\mathcal{C}$ and small category* $\mathbf{D}$*, there is a class of all limit cones for small diagrams of type* $\mathbf{D}$ *in $\mathcal{C}$, and a class of all limit cones for small diagrams of any shape in $\mathcal{C}$. The same holds for colimit cones and for various classes of them, such as limits over finite cones.*

Of course, some class of limit cones could be empty. There might be no limit cones in some case. But the class is well defined.

DEFINITION 6.25.

- *A* selection of limits *of shape* $\mathbf{D}$ *for a $\mathcal{U}$-category $\mathcal{C}$ is a locally small function from the class of diagrams of shape* $\mathbf{D}$ *in $\mathcal{C}$, to the class of limit cones, taking each diagram to a limit cone for it.*
- *A* selection of small limits *for a $\mathcal{U}$-category $\mathcal{C}$ is a locally small function from the class of small diagrams in $\mathcal{C}$, to the class of limit cones, taking each diagram to a limit cone for it.*

In these terms, the stronger part of Corollary 6.16 shows there is a definable selection of small limits for every sheaf topos, as well as a definable selection of small colimits. We call these the *canonical* limits and colimits for the sheaf topos. A great many other categories have naturally definable selections of limits and colimits, in a great many different ways, and we will certainly not attempt any systematic survey.

The class-theoretic reasoning so far in this section shows that MacClass suffices to prove these standard categorical results. There are many obvious analogues but these are the ones we will use:

THEOREM 6.26. *For any $\mathcal{U}$-category $\mathcal{C}$ and small category* $\mathbf{D}$:

(1) There is a $\mathcal{U}$-category $\mathcal{C}^{\mathbf{D}}$ of all small diagrams of shape $\mathbf{D}$ in $\mathcal{C}$, with natural transformations as arrows.
(2) Any selection of colimits of shape $\mathbf{D}$ in $\mathcal{C}$ defines a unique class functor $\varinjlim_{\mathbf{D}} : \mathcal{C}^{\mathbf{D}} \to \mathcal{C}$ taking each diagram to its selected colimit.
(3) Any selection of finite limits in $\mathcal{C}$ and any finite category $\mathbf{D}$ define a unique class functor $\varprojlim_{\mathbf{D}} : \mathcal{C}^{\mathbf{D}} \to \mathcal{C}$ taking each diagram to its selected limit.

LEMMA 6.27. *Given any $\mathcal{U}$-equivalence $f \dashv g : \mathcal{C} \to \mathcal{C}'$ and any small category $\mathbf{D}$, every selection of limits (or colimits) of shape $\mathbf{D}$ in $\mathcal{C}$ induces a selection in $\mathcal{C}'$.*

Proof. For any small $\delta : \mathbf{D} \to \mathcal{C}'$, take the selected limit cone of $f\delta : \mathbf{D} \to \mathcal{C}$ in $\mathcal{C}$. The adjoints of the arrows in that cone form a limit cone for δ in $\mathcal{C}'$. □

§7. Large-structure tools. The following theorems using a single universe in fact suffice for all the SGA. When SGA 4 invokes two universes $U \in V$, then V and all its subsets that are larger (or of higher rank) than U are just a shorthand in the familiar way proper classes are a shorthand in ZFC. See for example SGA on the Giraud Theorem (IV.1.2) and sheaf multilinear algebra (IV.10).

7.1. Grothendieck toposes. A *Grothendieck topos* is a category which has some $\mathcal{U}$-equivalence to a sheaf topos. This definition quantifies over $\mathcal{U}$-equivalences. So it is not set theoretic and MacClass cannot conclude there is a collection of all Grothendieck toposes.

A *geometric morphism* is a $\mathcal{U}$-adjunction $f^* \dashv f_* : \mathcal{E} \to \mathcal{E}'$ of Grothendieck toposes $\mathcal{E}, \mathcal{E}'$, where the left adjoint f^* is also left exact. The right adjoint f_* is the *direct image* functor, and f^* the *inverse image* functor. In particular, every $\mathcal{U}$-equivalence of Grothendieck toposes is a geometric morphism.

THEOREM 7.1. *Each Grothendieck topos $\mathcal{E}$ has a geometric morphism to $\mathcal{S}et$, unique up to natural isomorphism.*

Proof. Since $\mathcal{E}$ is $\mathcal{U}$-equivalent to a sheaf topos $\widetilde{\mathbf{C}}_J$, it suffices to prove the theorem for all $\widetilde{\mathbf{C}}_J$. Then there is at least the geometric morphism $\Delta \dashv \Gamma$, where the *global section functor* $\Gamma : \widetilde{C}_J \to \mathcal{S}et$ takes each sheaf $A \in \widetilde{C}_J$ to the set of arrows $1 \to A$. The left adjoint Δ, called the *discrete object functor*, takes each set S to the canonical S-fold coproduct of copies of 1 in $\widetilde{C}_J$. Uniqueness up to equivalence follows, since any inverse image functor has to take each set S to some sheaf isomorphic to that coproduct. □

For other examples, MacSet proves any continuous function $f : X \to X'$ between topological spaces induces suitable operations on sheaves and their transforms on those spaces. So MacClass proves f induces a geometric morphism $f^* \dashv f_*$ from the topos of sheaves $Top(X)$ to $Top(X')$, and given suitable separation conditions on the spaces every geometric morphism arises from a unique continuous function. See Mac Lane & Moerdijk (1992, p. 348).

LEMMA 7.2. *Every Grothendieck topos can be given a selection of all small limits, and one of all small colimits.*

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Proof. For any $\mathcal{U}$-equvalence $f^* \dashv f_*: \mathcal{E} \to \widetilde{\mathbf{C}}_J$, take the canonical selections on $\widetilde{\mathbf{C}}_J$, and transport them to $\mathcal{E}$ along the equivalence as in Lemma 6.27. □

Note the resulting selections on $\mathcal{E}$ are not uniquely defined. They depend not only on $\langle \mathbf{C}, J\rangle$ but on the adjunction $f^* \dashv f_*$.

7.2. *Giraud toposes.*

DEFINITION 7.3. *A* Giraud topos *is a $\mathcal{U}$-category $\mathcal{E}$ for which there exist*:

a) *a selection of limits for all finite diagrams*;
b) *a selection of coproducts for all sets of objects, and coproducts are stable disjoint unions;*
c) *a selection of quotients for all equivalence relation, and quotients are stable*;
d) *a small full subcategory* **G** *of generators with a set K* (*i.e., not just a class*) *of all jointly epimorphic sets of arrows between objects in* **G**.[7]

More briefly, a Giraud topos is a locally small elementary topos with a selected coproduct for each set of objects, and with a nice set of generators (Mac Lane & Moerdijk, 1992, p. 591). See that reference also for standard terms and results we assume here.

This definition quantifies over class selections, and so MacClass cannot conclude there is a collection of all Giraud toposes.

LEMMA 7.4. *Every Giraud topos $\mathcal{E}$ can be given a selection of all small colimits.* (*The case of all small limits will follow from Theorem 7.5.*)

Proof. Mac Lane & Moerdijk (1992, pp. 575–578) construct a coequalizer for each pair $f, g: A \to B$ in $\mathcal{E}$, using only finite limits, (infinitary) coproducts, and quotients for equivalence relations while Definition 7.3 says there exists a selection of each of those in $\mathcal{E}$. So there exists a selection of coequalizers. The usual specification of small colimits as coequalizers of small coproducts proves the lemma (Mac Lane, 1998, p. 113). □

THEOREM 7.5 (Giraud Theorem). *Every Giraud topos is a Grothendieck topos, and vice versa.*

Proof. §6.1 and §7.1 show every Grothendieck topos is a Giraud topos.

Conversely the proof by Mac Lane & Moerdijk (1992, pp. 578–587) works in MacClass to show every Giraud topos $\mathcal{E}$ is $\mathcal{U}$-equivalent to $\widetilde{\mathbf{G}}_J$, where **G** is any small full subcategory of generators as in the definition and J is the coverage of **G** by effective epimorphic sets of arrows.

Many of the steps are elementary category theory. Many concern only sets and work in MacSet. As to larger categorical issues, Lemma 7.4 lets us choose a single selection of all finite limits and of all small colimits for $\mathcal{E}$ to use throughout this proof. So we can take all constructions by finite limits and small colimits to have uniquely determined results.

So, let $\mathcal{E}$ be a Giraud topos with small full subcategory **G** of generators. Mac Lane and Moerdijk construct for each object E of $\mathcal{E}$ a presheaf $\underline{\mathrm{Hom}}_{\mathcal{E}}(A, E)$ on **G** by constructing the value $\underline{\mathrm{Hom}}_{\mathcal{E}}(A, E)(C)$ as a set of arrows in $\mathcal{E}$ for each **G** object C. Because $\mathcal{E}$ is locally small, this works in MacSet with one caveat: to show the presheaf exists in MacSet we

[7] In all concrete examples known to me K is provably a set since, given small **G**, some bounded MacSet formula defines K, as for the case of sheaves in Theorem 6.21.

need to fit the construction for all **G** objects C inside some one ambient set. This works by the method of ranks, like the proof of Theorem 3.2.

Once the construction is justified, Mac Lane and Moerdijk's verification of the properties of these presheaves works verbatim in MacSet.

Mac Lane and Moerdijk construct for each presheaf R on $\mathbf{G}$ a tensor product $R \otimes_{\mathbf{G}} A$ in $\mathcal{E}$ as a coequalizer of certain coproducts indexed by **G** arrows. These are small coproducts because **G** is small. These individual constructions yield a functor by using our chosen selection of colimits. And, echoing the remarks above, invoking our chosen selection of colimits (which is a class of course) does not mean quantifying over selections—or over any other classes.

Using the assumed selections of finite limits and small colimits, the constructions of $\underline{\mathrm{Hom}}_{\mathcal{E}}(A, E)$ and $R \otimes_{\mathbf{G}} A$ give functors

$$\mathcal{E} \xrightarrow{\underline{\mathrm{Hom}}_{\mathcal{E}}(A,\_)} \widehat{\mathbf{G}} \qquad\qquad \widehat{\mathbf{G}} \xrightarrow{\_\otimes_{\mathbf{G}}A} \mathcal{E}.$$

Mac Lane and Moerdijk define a coverage J on **G** by effective epimorphic sets of arrows. We justify this by using the set K required in the definition, clause (d). That is we define subset J such that $\langle C, S\rangle \in J$ iff S is an effective epimorphic set of arrows to C:

$$J \subseteq_0 G_0 \times K \ \subseteq_0 \ G_0 \times \mathcal{P}(G_1).$$

. Again, once MacClass shows these constructions are well defined, Mac Lane and Moerdijk's verification of their properties works verbatim in MacClass. The constructions give a $\mathcal{U}$-adjunction $\underline{\mathrm{Hom}}_{\mathcal{E}}(A, \_) \dashv (\_ \otimes_{\mathbf{G}} A)$. The verification that J is a coverage follows Mac Lane and Moerdijk's calculations verbatim, as does the verification that this $\mathcal{U}$-adjunction restricts to a $\mathcal{U}$-equivalence of $\mathcal{E}$ to $\widetilde{\mathbf{G}}_J$. □

***7.3. The 2-category* SitedTop.** The MacClass formulas defining Grothendieck toposes and Giraud toposes both quantify over classes. So MacClass cannot use those definitions to define a collection of all Grothendieck (or Giraud) toposes. MacClass plus a principle of class choice would prove that the Giraud definition, weakened to require only existence of finite limits and small coproducts without assuming selections of them, does define the class of Grothendieck toposes while quantifying only over sets. Indeed I cannot now prove that no set theoretic formula defines Grothendieck toposes in MacClass itself, though it seems unlikely.

MacClass does prove there is a collection of all sheaf toposes $\widetilde{\mathbf{C}}_J$ where each has a specified site $\langle \mathbf{C}, J\rangle$. And it proves there is a collection category of those, with geometric morphisms as arrows. Perhaps more interesting are these definitions:

DEFINITION 7.6.

- *A* sited *Grothendieck topos is a triple* $\langle \mathcal{E}, f^*, f_*\rangle$ *with*

$$f^* \dashv f_* \colon \mathcal{E} \to \widetilde{\mathbf{C}}_J$$

a $\mathcal{U}$*-equivalence of* $\mathcal{U}$*-category* $\mathcal{E}$ *to* $\widetilde{\mathbf{C}}_J$ *for some small site* $\langle \mathbf{C}, J\rangle$.
- *The collection category* SitedTop *has the collection of all sited Grothendieck toposes* $\langle \mathcal{E}, f^*, f_*\rangle$ *as objects. And a* SitedTop *arrow* g *from* $\langle \mathcal{E}_1, f_1^*, f_{1*}\rangle$ *to* $\langle \mathcal{E}_2, f_2^*, f_{2*}\rangle$ *is any geometric morphism*

$$g^* \dashv g_* \colon \mathcal{E}_1 \to \mathcal{E}_2.$$

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The arrows expressly ignore the site equivalences $f_1^* \dashv f_{1*}$ and $f_2^* \dashv f_{2*}$.

The point is that the definition of sited Grothendieck topos takes a $\mathcal{U}$-equivalence $f_1^* \dashv f_{1*}$ as an explicit parameter, and does not quantify over $\mathcal{U}$-equivalences. Then these equivalences are ignored by the arrows.

Indeed MacClass can define all natural transformations between geometric morphisms of sited Grothendieck toposes, quantifying only over sets. Thus it can define a collection 2-category of sited Grothendieck toposes, and geometric morphisms, and their natural transformations.

I have not yet pursued this any farther. Any interested reader could compare all this to the 2-category $\mathfrak{Top}$ of Grothendieck toposes, discussed by Johnstone (1977, p. 26) and much further in Johnstone (2002), using ZFC plus Grothendieck's version of universes as foundation.

7.4. Locally small sites. Compare small coverages in Definition 3.6.

DEFINITION 7.7. *A* class coverage $\mathcal{J} \subseteq C_0 \times \mathcal{P}_1(C_1)$ *on a class category $\mathcal{C}$ relates objects A to sets of arrows to A. The sets related to A, called $\mathcal{J}$-covering sets for A, must meet this condition: For any arrow $g\colon A \to B$ in $\mathcal{C}$ and $\mathcal{J}$-covering set S of B, there exists a $\mathcal{J}$-covering set S' of A such that for every $\mathcal{C}$ arrow $h \in S'$ the composite gh in $\mathcal{C}$ factors through some $\mathcal{C}$ arrow $f \in S$.*

We can use the power class $\mathcal{P}_1(C_1)$ because covering families for the large sites in general use are generated by sets of arrows. See Demazure & Grothendieck (1970, Exp. VI.4), Johnstone (2002, A.2.1.11), Milne (2016, p. 46f).[8]

Those and all other class sites in use are locally small, in other words $\mathcal{U}$-sites:

DEFINITION 7.8 (SGA 4.II.3.0.1). *A $\mathcal{U}$*-site $\langle \mathcal{C}, \mathcal{J}, \mathbf{D}\rangle$ *consists of*

(1) A coverage $\mathcal{J}$ on a $\mathcal{U}$-category $\mathcal{C}$, and

(2) a topological generating *small full subcategory $\mathbf{D}$ of $\mathcal{C}$: every $\mathcal{C}$ object A has some $\mathcal{J}$-covering set $\{g_i\colon D_i \to A | i \in I\}$ with all D_i objects of $\mathbf{D}$; and further for every $\mathcal{J}$-covering set $\{f_k\colon A_k \to A | k \in K\}$ in $\mathcal{C}$ there is at least one $\mathcal{J}$-covering set $\{g_i\colon D_i \to A | i \in I\}$ with all D_i objects of $\mathbf{D}$ such that each g_i factors through at least one of the f_k.*

(3) such that the intersection of $\mathcal{J}$ with $D_0 \times \mathcal{P}(D_1)$ is a set.[9]

Definition 7.8 implies the intersection of $\mathcal{J}$ with $D_0 \times \mathcal{P}(D_1)$ is a small coverage on $\mathbf{D}$, which we call the *induced coverage* J, so $\langle \mathbf{D}, J\rangle$ is a small site.

DEFINITION 7.9.

(1) A $\mathcal{U}$-presheaf is a contravariant $\mathcal{U}$-functor $\mathbf{F}\colon \mathcal{C} \to \mathcal{S}et$.

(2) A sheaf on $\mathcal{U}$-site $\langle \mathcal{C}, \mathcal{J}\rangle$ is a $\mathcal{U}$-presheaf on $\mathcal{C}$ with the sheaf property.

(3) Since the sheaf property on a $\mathcal{U}$-site quantifies only over sets, there is a collection category $\widetilde{\mathcal{C}}_{\mathcal{J}}$ of sheaves for each $\mathcal{U}$-site $\langle \mathcal{C}, \mathcal{J}, \mathbf{D}\rangle$.

The first point of these definitions is to yield:

---

[8] Shulman (2012, p. 2f) discusses general cardinality bounds on covering families.

[9] In concrete applications this holds since the $\mathcal{J}$-covering condition for any $\mathbf{D}$ object A is expressible by a bounded MacSet formula.

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LEMMA 7.10. *For a $\mathcal{U}$-site $\langle \mathcal{C}, \mathcal{J}, \mathbf{D} \rangle$, a presheaf on $\mathcal{C}$ is a $\mathcal{J}$-sheaf iff it has the sheaf property for all those $\mathcal{J}$-covers that have all their domains in* **D**.

LEMMA 7.11. *For any $\mathcal{U}$-site $\langle \mathcal{C}, \mathcal{J}, \mathbf{D} \rangle$, the functor $u_p \colon \widehat{\mathcal{C}} \to \widehat{\mathbf{D}}$ which restricts presheaves on $\mathcal{C}$ to presheaves on* **D** *takes every $\mathcal{J}$-sheaf to a J-sheaf. So u_p induces a functor on sheaves $u_s \colon \widetilde{\mathcal{C}}_{\mathcal{J}} \to \widetilde{\mathbf{D}}_J$.*

Since $\langle \mathbf{D}, J \rangle$ is a small site, $\widetilde{\mathbf{D}}_J$ is a Grothendieck topos, indeed a sheaf topos. The final point of these definitions is that u_s is half of an equivalence:

THEOREM 7.12 (Comparison Lemma). *Every $\mathcal{U}$-site $\langle \mathcal{C}, \mathcal{J}, \mathbf{D} \rangle$ has a definable (collection sized) adjoint equivalence $u^s \dashv u_s \colon \widetilde{\mathcal{C}}_{\mathcal{J}} \to \widetilde{\mathbf{D}}_J$.*

Proof. The chief point for MacClass is that the smallness conditions imply every $\mathcal{C}$ object A has a set of all arrows to it from **D** objects. Thus Johnstone's comma categories $(\mathcal{D} \downarrow U)$ (which we would write as $(\mathbf{D} \downarrow A)$) are small in MacClass, and the limit functors Johnstone uses to construct Kan extensions are well defined and locally small (2002, C2.2.3). Verdier's comma categories C/H are sets in MacClass, and his functors $u_!, u^*, u_*$ and u^s, u_s are well defined and locally small (SGA 4 III.4.1 p. 288ff). Similarly for Mac Lane & Moerdijk (1992, p. 588). As usual, the existence proofs for the various constructs in MacClass are more delicate than, for example, in ZFC. But with their existence proved, the standard published proofs of their properties work verbatim in MacClass. □

7.5. Duality and derived categories. The chief ideas of [Grothendieck duality] were known to me since 1959, but the lack of adequate foundations for homological algebra prevented me attempting a comprehensive revision. This gap in foundations is about to be filled by Verdier's dissertation, making a satisfactory presentation possible in principle. (Grothendieck quoted by Hartshorne, 1966, p. III)

Grothendieck (1957b) finds his duality theorem too limited. It was essentially as in Altman & Kleiman (1970): certain cohomology groups (and related groups) of nonsingular projective schemes are isomorphic in a natural way. The proof invokes proper class categories but really only uses sheaves and modules. It can be given in MacSet. Wiles (1995, p. 486) calls it "explicit duality over fields."

Grothendieck (1958, pp. 112–15) explains why duality should reach farther. By 1959 he believed the most unified and general tool is *derived categories*, now standard for Grothendieck duality. "Miraculously, the same formalism applies in étale cohomology, with quite different proofs" (Deligne, 1998, p. 17). Deligne uses them for étale Poincaré duality in SGA 4 XVII, XVIII and (Deligne, 1977).

Cohomology takes a module M on a scheme X and deletes nearly all its structure, highlighting just a little of it in the groups $H^n(X, M)$. The *derived category* $D(X)$ of modules on X deletes much of the same information but not all. Some manipulations work at this level which are obscured by excess detail at the level of modules and are impossible for lack of detail at the level of cohomology.

A scheme map $f \colon X \to Y$ sets up complicated relations between cohomology over X and Y. The effect on cohomology of a composite gf is not fully determined by the successive separate effects of f and then g (those determine it only up to a spectral sequence). A functor $Rf_* \colon D(X) \to D(Y)$ between derived categories approximates the effect of f on cohomology so that the approximation of successive effects is precisely the composite of the approximations:

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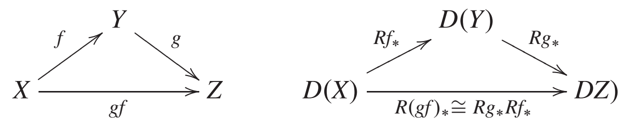

All variants of Grothendieck duality being developed today say the functor Rf_* has a right adjoint $Rf^!$, with further properties under some conditions on f.

The set theoretic issue is to form certain *categories of fractions*. In any small or class category $\mathbf{C}$ any class Σ of arrows can be made into isomorphisms by extending to a category $\mathbf{C}[\Sigma^{-1}]$ adding an inverse for each arrow in Σ while keeping the objects the same. In nice cases this uses a *calculus of fractions* so every arrow $A \to B$ in $\mathbf{C}[\Sigma^{-1}]$ is represented by a single pair of arrows in $\mathbf{C}$:

$$A \xleftarrow{\;s\;} C \xrightarrow{\quad f \quad} B \qquad s \in \Sigma$$

We define an equivalence relation on these pairs, and a composition rule so a pair $\langle s, f\rangle$ acts like a composite $fs^{-1}: A \to B$ even if $\mathbf{C}$ has no arrow s^{-1}. But when Σ is a proper class this can lead to collection-sized categories.

The derived category $D(X)$ starts with the category $\mathbf{K}(X)$ of complexes of quasi-coherent sheaves of modules over a scheme X, with *homotopy classes* of maps between complexes. Restricting attention to the quasi-coherent case does not affect the set theory involved in any obvious way, since it sets no bound on cardinality of the sheaves. The complexes and homotopy classes are sets, provably existing in MacSet. The derived category $D(X)$ is a certain calculus of fractions on $\mathcal{K}(X)$ (Eisenbud, 1995, p. 678ff).

Weibel (1994, p. 386) uses countable replacement to cut the classes of fractions down to sets for many important cases including modules on schemes. In ZFC proper classes of fractions actually do not exist since proper classes do not, and so ZFC cannot quantify over them or form collections of them. In MacClass, the classes of fractions always exist—as classes—with no limitation on the cases and of course no use of replacement.

Here the class category is $\mathbf{K}(X)$ and Σ is the class of *quasi-isomorphisms*, the homotopy classes inducing isomorphisms in all degrees of cohomology. For fixed A, B the relevant pairs are

$$A \xleftarrow{\;s\;} C \xrightarrow{\quad f \quad} B \qquad s \text{ any quasi-isomorphism}\ .$$

Each single equivalence class in $\mathbf{C}[\Sigma^{-1}]$ involves a proper class of pairs with different C. The collection $D(X)_1$ of arrows of $D(X)$ is the collection of these equivalence classes, while the class of objects is the class $D(X)_0 = \mathbf{K}(X)_0$ of complexes.

The key point both for working with derived categories mathematically and for formalizing them in MacClass is that the definition of $D(X)_1$ depends on infinitely many conditions on a proper class of complexes, but each single condition is small—it involves only sets. The definition gives a set theoretic abstract. The graphs of domain, codomain, and composition functors are similar. MacClass proves there is a derived category $D(X)$, with a class of objects and collection of arrows.

So current work on Grothendieck duality is formalizable in MacClass. For debate over mathematical strategies (not logical foundations) see Conrad (2000, preface), Lipman in (Lipman & Hashimoto, 2009, pp. 7–9), and Neeman (2010, pp. 294–300). Hartshorne (1966, pp. 1–13) describes an “ideal form” of the theorem and suggests “Perhaps some day this type of construction will be done more elegantly using the language of fibred categories and results of Giraud’s thesis” (p. 16).

7.6. Fibred categories. Universes first appeared in print in SGA 1 VI on fibred categories, which are a way to treat a class or category of categories as a single category. So SGA 4 VI calculates limits of families of Grothendieck toposes by using fibred toposes. In much of SGA 4 fibred toposes are presented by fibred sites. The logical issues are essentially the same as in §7.4. Many applications can be cast in MacSet in terms of sites, while the general facts are clearer and more concise in MacClass using toposes and fibred families of them.

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BIBLIOGRAPHY

The abbreviation SGA 1 refers to Grothendieck (1971) and SGA 4 is Artin et al. (1972). EGA I and IV are respectively Grothendieck & Dieudonné (1971) and Grothendieck & Dieudonné (1964).

Altman, A. & Kleiman, S. (1970). *An Introduction to Grothendieck Duality Theory*. Springer Lecture Notes in Mathematics, Vol. 146. New York: Springer-Verlag.
Artin, M. (1962). *Grothendieck Topologies*: *Notes on a Seminar*. Cambridge, MA: Harvard University, Department of Mathematics.
Artin, M., Grothendieck, A., & Verdier, J.-L. (1972). *Théorie des Topos et Cohomologie Etale des Schémas*. Séminaire de géométrie algébrique du Bois-Marie, Vol. 4. New York: Springer-Verlag. Three volumes, cited as SGA 4.
Baer, R. (1940). Abelian groups that are direct summands of every containing abelian group. *Bulletin of the American Mathematical Society*, **46**, 800–806.
Barr, M. (1974). Toposes without points. *Journal of Pure and Applied Algebra*, **5**, 265–280.
Blass, A. (1979). Injectivity, projectivity, and the axiom of choice. *Transactions of the American Mathematical Society*, **255**, 31–59.
Conrad, B. (2000). *Grothendieck Duality and Base Charge*. Lecture Notes in Mathematics, Vol. 1750. New York: Springer-Verlag.
Deligne, P. (1977). Cohomologie étale: les points de départ. In Deligne, P., editor. *Cohomologie Étale*. New York: Springer-Verlag, pp. 4–75.
Deligne, P. (1998). Quelques idées maîtresses de l'œuvre de A. Grothendieck. *Matériaux pour l'Histoire des Mathématiques au XX*e *Siècle* (*Nice, 1996*). Paris: Société mathématique de France, pp. 11–19.
Demazure, M. & Grothendieck, A. (1970). *Schémas en groupes*. Séminaire de géométrie algébrique du Bois-Marie, Vol. 3. New York: Springer-Verlag.
Eckmann, B. & Schopf, A. (1953). Über injektive Moduln. *Archiv der Mathematik*, **4**(2), 75–78.
Eisenbud, D. (1995). *Commutative Algebra*. New York: Springer-Verlag.
Grothendieck, A. (1957a). Sur quelques points d'algèbre homologique. *Tôhoku Mathematical Journal*, **9**, 119–221.

https://doi.org/10.1017/S1755020319000340 Published online by Cambridge University Press

Grothendieck, A. (1957b). Théorèmes de dualité pour les faisceaux algébriques cohérents, exposé 149. *Séminaire Bourbaki*. Secrétariat mathématique. Paris: Université Paris, pp. 169–193.

Grothendieck, A. (1958). The cohomology theory of abstract algebraic varieties. *Proceedings of the International Congress of Mathematicians, 1958*. Cambridge: Cambridge University Press, pp. 103–118.

Grothendieck, A. (1971). *Revêtements Étales et Groupe Fondamental*. Séminaire de géométrie algébrique du Bois-Marie, Vol. 1. New York: Springer-Verlag. Cited as SGA 1.

Grothendieck, A. & Dieudonné, J. (1964). *Éléments de Géométrie Algébrique IV*: *Étude locale des schémas et des morphismes de schémas, Première partie*. Publications Mathématiques, Vol. 20. Paris: Institut des Hautes Études Scientifiques.

Grothendieck, A. & Dieudonné, J. (1971). *Éléments de Géométrie Algébrique I*. New York: Springer-Verlag.

Hartshorne, R. (1966). *Residues and Duality, Lecture Notes of a Seminar on the Work of A. Grothendieck given at Harvard 1963–64*. Lecture Notes in Mathematics, Vol. 20. New York: Springer-Verlag.

Hartshorne, R. (1977). *Algebraic Geometry*. New York: Springer-Verlag.

Johnstone, P. (1977). *Topos Theory*. London: Academic Press.

Johnstone, P. (2002). *Sketches of an Elephant*: *A Topos Theory Compendium*. Oxford: Oxford University Press. To be finished as three volumes.

Kunen, K. (1983). *Set Theory*: *An Introduction to Independence Proofs*. Amsterdam: North-Holland.

Lawvere, F. W. (1965). An elementary theory of the category of sets. Lecture notes of the Department of Mathematics, University of Chicago. Reprint with commentary by the author and Colin McLarty in: *Reprints in Theory and Applications of Categories*, No. 11 (2005) pp. 1–35, on-line at http://138.73.27.39/tac/reprints/articles/11/tr11abs.html.

Lipman, J. & Hashimoto, M. (2009). *Foundations of Grothendieck Duality for Diagrams of Schemes*. New York: Springer-Verlag.

Mac Lane, S. (1998). *Categories for the Working Mathematician* (second edition). New York: Springer-Verlag.

Mac Lane, S. & Moerdijk, I. (1992). *Sheaves in Geometry and Logic*. New York: Springer-Verlag.

Mathias, A. R. D. (2001). The strength of Mac Lane set theory. *Annals of Pure and Applied Logic*, **110**, 107–234.

McLarty, C. (2006). Two aspects of constructivism in category theory. *Philosophia Scientiae*, **6**, 95–114. Cahier Spécial.

Milne, J. (2016). *Étale Cohomology*. Princeton, NJ: Princeton University Press.

Neeman, A. (2010). Derived categories and Grothendieck duality. In Holm, T., Jørgensen, P., and Rouquier, R., editors. *Triangulated Categories*. Cambridge: Cambridge University Press, pp. 290–350.

Shulman, M. (2012). Exact completions and small sheaves. Preprint on the mathematics arXiv, arXiv:1203.4318v2.

Takeuti, G. (1978). A conservative extension of Peano Arithmetic. *Two Applications of Logic to Mathematics*. Princeton, NJ: Princeton University Press, pp. 77–135.

Takeuti, G. (1987). *Proof Theory* (second edition). New York: Elsevier Science Ltd.

Tamme, G. (1994). *Introduction to Etale Cohomology*. New York: Springer-Verlag.

van Osdol, D. (1975). Homological algebra in topoi. *Proceedings of the American Mathematical Society*, **50**, 52–54.

Weibel, C. (1994). *An Introduction to Homological Algebra*. Cambridge: Cambridge University Press.

Wiles, A. (1995). Modular elliptic curves and Fermat's Last Theorem. *Annals of Mathematics*, **141**, 443–551.

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