CANTOR'S THEOREM MAY FAIL FOR FINITARY PARTITIONS

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Abstract. A partition is finitary if all its members are finite. For a set A, $\mathscr{B}(A)$ denotes the set of all finitary partitions of A. It is shown consistent with ZF (without the axiom of choice) that there exist an infinite set A and a surjection from A onto $\mathscr{B}(A)$. On the other hand, we prove in ZF some theorems concerning $\mathscr{B}(A)$ for infinite sets A, among which are the following:

(1) If there is a finitary partition of A without singleton blocks, then there are no surjections from A onto $\mathscr{B}(A)$ and no finite-to-one functions from $\mathscr{B}(A)$ to A.
(2) For all $n \in \omega$, $|A^n| < |\mathscr{B}(A)|$.
(3) $|\mathscr{B}(A)| \neq |\mathrm{seq}(A)|$, where $\mathrm{seq}(A)$ is the set of all finite sequences of elements of A.

§1. Introduction. In 1891, Cantor [1] proved that, for all sets A, there are no surjections from A onto $\mathscr{P}(A)$ (the power set of A). Under the axiom of choice, for infinite sets A, several sets related to A have the same cardinality as $\mathscr{P}(A)$ or A; for example, $\mathcal{S}(A)$ (the set of all permutations of A) and $\mathrm{Part}(A)$ (the set of all partitions of A) have the same cardinality as $\mathscr{P}(A)$, and A^2, $\mathrm{fin}(A)$ (the set of all finite subsets of A), $\mathrm{seq}(A)$ (the set of all finite sequences of elements of A), and $\mathrm{seq}^{1\text{-}1}(A)$ (the set of all finite sequences without repetition of elements of A) have the same cardinality as A. However, without the axiom of choice, this is no longer the case. In 1924, Tarski [19] proved that the statement that A^2 has the same cardinality as A for all infinite sets A is in fact equivalent to the axiom of choice.

Over the past century, various variations of Cantor's theorem have been investigated in ZF (the Zermelo–Fraenkel set theory without the axiom of choice), with A or $\mathscr{P}(A)$ replaced by a set which has the same cardinality under the axiom of choice. Specker [18] proves that, for all infinite sets A, there are no injections from $\mathscr{P}(A)$ into A^2. Halbeisen and Shelah [5] prove that $|\mathrm{fin}(A)| < |\mathscr{P}(A)|$ and $|\mathrm{seq}^{1\text{-}1}(A)| \neq |\mathscr{P}(A)| \neq |\mathrm{seq}(A)|$. Forster [3] proves that there are no finite-to-one functions from $\mathscr{P}(A)$ to A. Recently, Peng and Shen [8] prove that there are no surjections from $\omega \times A$ onto $\mathscr{P}(A)$, and Peng, Shen, and Wu [9] prove that the existence of an infinite set A and a surjection from A^2 onto $\mathscr{P}(A)$ is consistent with ZF. The variations of Cantor's theorem with $\mathscr{P}(A)$ replaced by $\mathcal{S}(A)$ are investigated in [2, 15–17].

For a set A, let $\mathscr{B}(A)$ be the set of all finitary partitions of A, where a partition is finitary if all its members are finite. We use the symbol $\mathscr{B}$ to denote this notion just because $|\mathscr{B}(n)|$ is the nth Bell number. The axiom of choice implies that $\mathscr{B}(A)$ and $\mathscr{P}(A)$ have the same cardinality for infinite sets A, but each of "$|\mathscr{B}(A)| < |\mathscr{P}(A)|$",

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"$|\mathscr{P}(A)| < |\mathscr{B}(A)|$", and "$|\mathscr{B}(A)|$ and $|\mathscr{P}(A)|$ are incomparable" for some infinite set A is consistent with ZF. Recently, Phansamdaeng and Vejjajiva [10] prove that $|\text{fin}(A)| < |\mathscr{B}(A)|$ for all infinite sets A.

In this paper, we further study the variations of Cantor's theorem with $\mathscr{P}(A)$ replaced by $\mathscr{B}(A)$. We show that Cantor's theorem may fail for finitary partitions in the sense that the existence of an infinite set A and a surjection from A onto $\mathscr{B}(A)$ is consistent with ZF. Nevertheless, we prove in ZF some theorems concerning $\mathscr{B}(A)$ for infinite sets A, among which are the following:

(1) If there is a finitary partition of A without singleton blocks, then there are no surjections from A onto $\mathscr{B}(A)$ and no finite-to-one functions from $\mathscr{B}(A)$ to A.
(2) For all $n \in \omega$, $|A^n| < |\mathscr{B}(A)|$.
(3) $|\mathscr{B}(A)| \neq |\text{seq}(A)|$.

§2. Some notation and preliminary results. Throughout this paper, we shall work in ZF. In this section, we indicate briefly our use of some terminology and notation. For a function f, we use $\text{dom}(f)$ for the domain of f, $\text{ran}(f)$ for the range of f, $f[A]$ for the image of A under f, $f^{-1}[A]$ for the inverse image of A under f, and $f{\upharpoonright}A$ for the restriction of f to A. For functions f and g, we use $g \circ f$ for the composition of g and f. We write $f : A \to B$ to express that f is a function from A to B, and $f : A \twoheadrightarrow B$ to express that f is a function from A *onto* B. For a set A, $|A|$ denotes the cardinality of A.

DEFINITION 2.1. Let A, B be arbitrary sets.

(1) $|A| = |B|$, or $A \approx B$, if there is a bijection between A and B.
(2) $|A| \leqslant |B|$, or $A \preccurlyeq B$, if there is an injection from A into B.
(3) $|A| \leqslant^* |B|$, or $A \preccurlyeq^* B$, if there is a surjection from a subset of B onto A.
(4) $|A| < |B|$ if $|A| \leqslant |B|$ and $|A| \neq |B|$.

Clearly, if $A \preccurlyeq B$ then $A \preccurlyeq^* B$, and if $A \preccurlyeq^* B$ then $\mathscr{P}(A) \preccurlyeq \mathscr{P}(B)$.

In the sequel, we shall frequently use expressions like "one can explicitly define" in our formulations, which is illustrated by the following example.

THEOREM 2.2 (Cantor–Bernstein). *From injections $f : A \to B$ and $g : B \to A$, one can explicitly define a bijection $h : A \to B$.*

PROOF. See [4, Theorem 3.14]. ⊣

Formally, Theorem 2.2 states that one can define a class function H without free variables such that, whenever f is an injection from A into B and g is an injection from B into A, $H(f, g)$ is defined and is a bijection between A and B. Consequently, if $|A| \leqslant |B|$ and $|B| \leqslant |A|$, then $|A| = |B|$.

DEFINITION 2.3. Let A be a set and let f be a function.

(1) A is *Dedekind infinite* if $\omega \preccurlyeq A$; otherwise, A is *Dedekind finite*.
(2) A is *power Dedekind infinite* if $\mathscr{P}(A)$ is Dedekind infinite; otherwise, A is *power Dedekind finite*.
(3) f is *(Dedekind) finite-to-one* if for every $z \in \text{ran}(f)$, $f^{-1}[\{z\}]$ is (Dedekind) finite.

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Clearly, if f and g are (Dedekind) finite-to-one functions, so is $g \circ f$ (cf. [12, Fact 2.8]). It is well known that A is Dedekind infinite if and only if there exists a bijection between A and a proper subset of A. For power Dedekind infinite sets, recall Kuratowski's celebrated theorem.

Theorem 2.4 (Kuratowski). *A is power Dedekind infinite if and only if $\omega \preccurlyeq^* A$.*

Proof. See [4, Proposition 5.4]. ⊣

The following two facts are Corollaries 2.9 and 2.11 of [12], respectively.

Fact 2.5. *If A is power Dedekind infinite and there exists a finite-to-one function from A to B, then B is power Dedekind infinite.*

Fact 2.6. *If A^n is power Dedekind infinite, so is A.*

Definition 2.7. Let P be a partition of A. We say that P is *finitary* if all blocks of P are finite, and write $\mathrm{ns}(P)$ for the set of non-singleton blocks of P. For $x \in A$, we write $[x]_P$ for the unique block of P which contains x. The equivalence relation $\sim_P$ on A induced by P is defined by

$$x \sim_P y \qquad \text{if and only if} \qquad [x]_P = [y]_P.$$

Definition 2.8. Let A be an arbitrary set.

(1) $\mathrm{Part}(A)$ is the set of all partitions of A.
(2) $\mathrm{Part_{fin}}(A) = \{P \in \mathrm{Part}(A) \mid P \text{ is finite}\}$.
(3) $\mathscr{B}(A) = \{P \in \mathrm{Part}(A) \mid P \text{ is finitary}\}$.
(4) $\mathscr{B}_{\mathrm{fin}}(A) = \{P \in \mathscr{B}(A) \mid \mathrm{ns}(P) \text{ is finite}\}$.
(5) $\mathrm{fin}(A)$ is the set of all finite subsets of A.
(6) $\mathrm{seq}(A) = \{f \mid f \text{ is a function from an } n \in \omega \text{ to } A\}$.
(7) $\mathrm{seq}^{1\text{-}1}(A) = \{f \mid f \text{ is an injection from an } n \in \omega \text{ into } A\}$.

Below we list some basic relations between the cardinalities of these sets. We first note that $\mathrm{fin}(A) \preccurlyeq^* \mathrm{seq}^{1\text{-}1}(A) \preccurlyeq \mathrm{seq}(A)$. The next three facts are Facts 2.13, 2.16, and 2.17 of [12], respectively.

Fact 2.9. *If A is infinite, then $\mathrm{fin}(A)$ and $\mathscr{P}(A)$ are power Dedekind infinite.*

Fact 2.10. $\mathrm{seq}^{1\text{-}1}(A) \preccurlyeq \mathrm{fin}(\mathrm{fin}(A))$.

Fact 2.11. *There is a finite-to-one function from $\mathrm{fin}(\mathrm{fin}(A))$ to $\mathrm{fin}(A)$.*

The next three facts are Facts 2.19, 2.20, and Corollary 2.23 of [15], respectively.

Fact 2.12. *If A is non-empty, then $\mathrm{seq}(A)$ is Dedekind infinite.*

Fact 2.13. *If A is Dedekind finite, then there is a Dedekind finite-to-one function from $\mathrm{seq}(A)$ to ω.*

Fact 2.14. *If A is Dedekind infinite, then $\mathrm{seq}(A) \approx \mathrm{seq}^{1\text{-}1}(A)$.*

Fact 2.15. $\mathrm{Part}(A) \preccurlyeq \mathscr{P}(A^2)$.

Proof. The function that maps each partition P of A to $\sim_P$ is an injection from $\mathrm{Part}(A)$ into $\mathscr{P}(A^2)$. ⊣

Corollary 2.16. *If A is power Dedekind finite, then* $\mathrm{Part}(A)$, *and hence also* $\mathrm{Part}_{\mathrm{fin}}(A)$ *and* $\mathscr{B}(A)$, *are Dedekind finite.*

Proof. If A is power Dedekind finite, so is A^2 by Fact 2.6, and thus $\mathscr{P}(A^2)$ is Dedekind finite, which implies that also $\mathrm{Part}(A)$, $\mathrm{Part}_{\mathrm{fin}}(A)$ and $\mathscr{B}(A)$ are Dedekind finite by Fact 2.15. ⊣

Fact 2.17. $\mathscr{B}_{\mathrm{fin}}(A) \preccurlyeq \mathrm{fin}(\mathrm{fin}(A))$.

Proof. The function that maps each $P \in \mathscr{B}_{\mathrm{fin}}(A)$ to $\mathrm{ns}(P)$ is an injection from $\mathscr{B}_{\mathrm{fin}}(A)$ into $\mathrm{fin}(\mathrm{fin}(A))$. ⊣

Fact 2.18. $\mathscr{B}_{\mathrm{fin}}(A) \preccurlyeq \mathrm{Part}_{\mathrm{fin}}(A)$.

Proof. If A is finite, then $\mathscr{B}_{\mathrm{fin}}(A) = \mathrm{Part}_{\mathrm{fin}}(A)$; otherwise, the function that maps each $P \in \mathscr{B}_{\mathrm{fin}}(A)$ to $\mathrm{ns}(P) \cup \{\bigcup(P \setminus \mathrm{ns}(P))\}$ is an injection from $\mathscr{B}_{\mathrm{fin}}(A)$ into $\mathrm{Part}_{\mathrm{fin}}(A)$. ⊣

Fact 2.19. *If* $|A| \geqslant 5$, *then* $\mathrm{fin}(A) \preccurlyeq \mathscr{B}_{\mathrm{fin}}(A)$ *and* $\mathscr{P}(A) \preccurlyeq \mathrm{Part}_{\mathrm{fin}}(A)$.

Proof. Let $E = \{a, b, c, d, e\}$ be a 5-element subset of A. We define functions $f : \mathrm{fin}(A) \to \mathscr{B}_{\mathrm{fin}}(A)$ and $g : \mathscr{P}(A) \to \mathrm{Part}_{\mathrm{fin}}(A)$ by setting, for $B \in \mathrm{fin}(A)$ and $C \in \mathscr{P}(A)$,

$$f(B) = \begin{cases} (\{B\} \cup [A \setminus B]^1) \setminus \{\varnothing\}, & \text{if } B \text{is not a singleton},\\ \{\{a, x\}, E \setminus \{a, x\}\} \cup [A \setminus (B \cup E)]^1, & \text{if } B = \{x\} \text{ for some } x \neq a,\\ \{\{a\}, \{b, c\}, \{d, e\}\} \cup [A \setminus E]^1, & \text{if } B = \{a\}, \end{cases}$$

and

$$g(C) = \begin{cases} \{C, A \setminus C\} \setminus \{\varnothing\}, & \text{if } a \notin C,\\ (\{C, A \setminus (C \cup E)\} \cup [E \setminus C]^1) \setminus \{\varnothing\}, & \text{if } a \in C, |C \cap E| \leqslant 3,\\ \{C \setminus \{a\}, \{a\}, E \setminus C, A \setminus (C \cup E)\} \setminus \{\varnothing\}, & \text{if } a \in C, |C \cap E| = 4,\\ \{C \setminus \{b, c, d, e\}, \{b, c\}, \{d, e\}, A \setminus C\} \setminus \{\varnothing\}, & \text{if } E \subseteq C, \end{cases}$$

where $[D]^1$ denotes the set of 1-element subsets of D. It is easy to see that f and g are injective. ⊣

The following corollary immediately follows from Facts 2.9 and 2.19.

Corollary 2.20. *If A is infinite, then $\mathscr{B}_{\mathrm{fin}}(A)$ and $\mathscr{B}(A)$ are power Dedekind infinite.*

For infinite sets A, the relations between the cardinalities of $\mathrm{fin}(A)$, $\mathscr{P}(A)$, $\mathscr{B}_{\mathrm{fin}}(A)$, $\mathscr{B}(A)$, $\mathrm{Part}_{\mathrm{fin}}(A)$, $\mathrm{Part}(A)$, and $\mathscr{P}(A^2)$ can be visualized by Figure 1 (where for two sets X and Y, $X \longrightarrow Y$ means $|X| \leqslant |Y|$, $X \dashrightarrow Y$ means $|X| < |Y|$, and X - - - Y means $|X| \neq |Y|$).

In Figure 1, the $\leqslant$-relations have already been established, and the inequalities $|\mathrm{fin}(A)| < |\mathscr{P}(A)|$ and $|\mathrm{fin}(A)| < |\mathscr{B}(A)|$ are proved in [5, Theorem 3] and [10, Theorem 3.7], respectively. The inequalities $|\mathscr{B}_{\mathrm{fin}}(A)| < |\mathrm{Part}_{\mathrm{fin}}(A)|$ and $|\mathscr{B}_{\mathrm{fin}}(A)| \neq |\mathscr{P}(A)|$ will be proved in Section 4. The other relations not indicted in the figure cannot be proved in ZF, as shown in the next section.

§3. Permutation models and consistency results. We refer the readers to [4, Chapter 8] or [6, Chapter 4] for an introduction to the theory of permutation models. Permutation models are not models of ZF; they are models of ZFA (the Zermelo–Fraenkel set theory with atoms). Nevertheless, they indirectly give, via the Jech–Sochor theorem (cf. [4, Theorem 17.2] or [6, Theorem 6.1]), models of ZF.

Let A be the set of atoms, let $\mathcal{G}$ be a group of permutations of A, and let $\mathfrak{F}$ be a normal filter on $\mathcal{G}$. We write $\mathrm{sym}_{\mathcal{G}}(x)$ for the set $\{\pi \in \mathcal{G} \mid \pi x = x\}$, where $\pi \in \mathcal{G}$ extends to a permutation of the universe by

$$\pi x = \{\pi y \mid y \in x\}.$$

Then x belongs to the permutation model $\mathcal{V}$ determined by $\mathcal{G}$ and $\mathfrak{F}$ if and only if $x \subseteq \mathcal{V}$ and $\mathrm{sym}_{\mathcal{G}}(x) \in \mathfrak{F}$.

For each $E \subseteq A$, we write $\mathrm{fix}_{\mathcal{G}}(E)$ for the set $\{\pi \in \mathcal{G} \mid \forall a \in E(\pi a = a)\}$. Let $\mathcal{I} \subseteq \mathcal{P}(A)$ be a normal ideal and let $\mathfrak{F}$ be the normal filter on $\mathcal{G}$ generated by the subgroups $\{\mathrm{fix}_{\mathcal{G}}(E) \mid E \in \mathcal{I}\}$. Then x belongs to the permutation model $\mathcal{V}$ determined by $\mathcal{G}$ and $\mathcal{I}$ if and only if $x \subseteq \mathcal{V}$ and there exists an $E \in \mathcal{I}$ such that $\mathrm{fix}_{\mathcal{G}}(E) \subseteq \mathrm{sym}_{\mathcal{G}}(x)$; that is, every $\pi \in \mathcal{G}$ fixing E pointwise also fixes x. Such an E is called a *support* of x.

3.1. A model for $|\mathscr{B}(A)| \leqslant^* |A|$ and $|\mathscr{B}(A)| < |\mathcal{P}(A)|$. We construct a permutation model $\mathcal{V}_{\mathscr{B}}$ in which the set A of atoms satisfies $|\mathscr{B}(A)| \leqslant^* |A|$ and $|\mathscr{B}(A)| < |\mathcal{P}(A)|$. The atoms are constructed by recursion as follows:

(i) $A_0 = \varnothing$ and $\mathcal{G}_0 = \{\varnothing\}$ is the group of all permutations of A_0.
(ii) $A_{n+1} = A_n \cup \{(n, P, k) \mid P \in \mathscr{B}(A_n) \text{ and } k \in \omega\}$.
(iii) $\mathcal{G}_{n+1}$ is the group of permutations of A_{n+1} consisting of all permutations h for which there exists a $g \in \mathcal{G}_n$ such that:
   - $g = h{\restriction}A_n$;
   - for each $P \in \mathscr{B}(A_n)$, there exists a permutation q of ω such that $h(n, P, k) = (n, \{g[D] \mid D \in P\}, q(k))$ for all $k \in \omega$.

Let $A = \bigcup_{n\in\omega} A_n$ be the set of atoms, let $\mathcal{G}$ be the group of permutations of A consisting of all permutations π such that $\pi{\restriction}A_n \in \mathcal{G}_n$ for all $n \in \omega$, and let $\mathfrak{F}$ be the normal filter on $\mathcal{G}$ generated by the subgroups $\{\mathrm{fix}_{\mathcal{G}}(A_n) \mid n \in \omega\}$. The permutation model determined by $\mathcal{G}$ and $\mathfrak{F}$ is denoted by $\mathcal{V}_{\mathscr{B}}$.

LEMMA 3.1. *For every $P \in \mathscr{B}(A)$, $P \in \mathcal{V}_{\mathscr{B}}$ if and only if $\mathrm{ns}(P) \subseteq \mathcal{P}(A_m)$ for some $m \in \omega$.*

PROOF. Let $P \in \mathscr{B}(A)$. If $\mathrm{ns}(P) \subseteq \mathcal{P}(A_m)$ for some $m \in \omega$, then clearly $\mathrm{fix}_{\mathcal{G}}(A_m) \subseteq \mathrm{sym}_{\mathcal{G}}(P)$, which implies that $P \in \mathcal{V}_{\mathscr{B}}$. For the other direction, suppose $P \in \mathcal{V}_{\mathscr{B}}$ and let $m \in \omega$ be such that $\mathrm{fix}_{\mathcal{G}}(A_m) \subseteq \mathrm{sym}_{\mathcal{G}}(P)$; that is, every $\pi \in \mathcal{G}$ fixing A_m pointwise also fixes P. We claim $\mathrm{ns}(P) \subseteq \mathcal{P}(A_m)$. Assume towards a contradiction that $x \sim_P y$ for some distinct x, y such that one of x and y is not in A_m. Suppose that $x = (n, Q, k)$ and $y = (n', Q', k')$, and assume without loss of generality $n' \leqslant n$. Then $x \notin A_m$ and thus $m \leqslant n$. Let $l \in \omega$ be such that $(n, Q, l) \notin [y]_P$ and let q be the transposition that swaps k and l. Since P is finitary, such an l exists. Let h be the permutation of A_{n+1} such that h fixes A_n pointwise and for all $R \in \mathscr{B}(A_n)$ and all $j \in \omega$, $h(n, R, j) = (n, R, q(j))$ if $R = Q$, and $h(n, R, j) = (n, R, j)$ otherwise. Then $h \in \mathcal{G}_{n+1}$ fixes $A_{n+1} \setminus \{x, (n, Q, l)\}$

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pointwise. Hence $h(y) = y$. Extend h in a straightforward way to some $\pi \in \mathcal{G}$. Then $\pi \in \mathrm{fix}_{\mathcal{G}}(A_m \cup \{y\})$ and $\pi(x) = (n, Q, l) \notin [y]_P$. Thus π moves P, which is a contradiction. ⊣

LEMMA 3.2. *In* $\mathcal{V}_{\mathcal{B}}$, $|\mathcal{B}(A)| \leqslant^* |A|$ *and* $|\mathcal{B}(A)| < |\mathcal{P}(A)|$.

PROOF. Let Φ be the function on $\{P \in \mathcal{B}(A) \mid \exists m \in \omega(\mathrm{ns}(P) \subseteq \mathcal{P}(A_m))\}$ defined by

$$\Phi(P) = \{(n_P, P \cap \mathcal{P}(A_{n_P}), k) \mid k \in \omega\},$$

where n_P is the least $m \in \omega$ such that $\mathrm{ns}(P) \subseteq \mathcal{P}(A_m)$. Clearly, $\Phi \in \mathcal{V}_{\mathcal{B}}$. In $\mathcal{V}_{\mathcal{B}}$, by Lemma 3.1, Φ is an injection from $\mathcal{B}(A)$ into $\mathcal{P}(A)$, and the sets in the range of Φ are pairwise disjoint, which implies that $|\mathcal{B}(A)| \leqslant^* |A|$. Since $|\mathcal{P}(A)| \nleqslant^* |A|$ by Cantor's theorem, it follows that $|\mathcal{B}(A)| < |\mathcal{P}(A)|$. ⊣

Now the next theorem immediately follows from Lemma 3.2 and the Jech–Sochor theorem.

THEOREM 3.3. *It is consistent with* ZF *that there exists an infinite set A for which* $|\mathcal{B}(A)| \leqslant^* |A|$ *and* $|\mathcal{B}(A)| < |\mathcal{P}(A)|$.

3.2. A model for $|\mathcal{P}(A)| < |\mathcal{B}(A)| < |\mathrm{Part}_{\mathrm{fin}}(A)|$. We show that the ordered Mostowski model $\mathcal{V}_{\mathrm{M}}$ (cf. [4, pp. 198–202] or [6, Section 4.5]) is a model of this kind. Recall that the set A of atoms carries an ordering $<_{\mathrm{M}}$ which is isomorphic to the ordering of the rational numbers, the permutation group $\mathcal{G}$ consists of all automorphisms of $\langle A, <_{\mathrm{M}}\rangle$, and $\mathcal{V}_{\mathrm{M}}$ is determined by $\mathcal{G}$ and finite supports. Clearly, the ordering $<_{\mathrm{M}}$ belongs to $\mathcal{V}_{\mathrm{M}}$ (cf. [4, Lemma 8.10]). In $\mathcal{V}_{\mathrm{M}}$, A is infinite but power Dedekind finite (cf. [4, Lemma 8.13]), and thus, by Fact 2.6 and Corollary 2.16, $\mathcal{P}(A^2)$ and $\mathcal{B}(A)$ are Dedekind finite.

LEMMA 3.4. *In* $\mathcal{V}_{\mathrm{M}}$, $\mathcal{B}(A) = \mathcal{B}_{\mathrm{fin}}(A)$.

PROOF. Let $P \in \mathcal{V}_{\mathrm{M}}$ be a finitary partition of A and let E be a finite support of P. We claim $\mathrm{ns}(P) \subseteq \mathcal{P}(E)$. Assume towards a contradiction that $x \sim_P y$ for some distinct x, y such that $x \notin E$. Since P is finitary, we can find a $\pi \in \mathrm{fix}_{\mathcal{G}}(E \cup \{y\})$ such that $\pi(x) \notin [y]_P$. Hence π moves P, contradicting that E is a support of P. Thus $\mathrm{ns}(P) \subseteq \mathcal{P}(E)$, so $P \in \mathcal{B}_{\mathrm{fin}}(A)$. ⊣

The next two lemmas are Lemmas 8.11(b) and 8.12 of [4], respectively.

LEMMA 3.5. *Every $x \in \mathcal{V}_{\mathrm{M}}$ has a least support.*

LEMMA 3.6. *If E is an n-element subset of A, then E supports exactly 2^{2n+1} subsets of A.*

LEMMA 3.7. *For each $n \in \omega$, let $B_n^\star$ be the number of partitions of n without singleton blocks*; *that is,*

$$B_n^\star = |\{P \in \mathcal{B}(n) \mid \mathrm{ns}(P) = P\}|.$$

If $n \geqslant 23$, then $2^{2n+2} < B_n^\star$.

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Proof. For each $n \in \omega$, let B_n be the nth Bell number; that is, $B_n = |\mathscr{B}(n)|$. Recall Dobinski's formula (see, for example, [11]):

$$B_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}.$$

It is easy to see that $B_n = B_n^\star + B_{n+1}^\star$. Hence, for $n \geqslant 23$, we have

$$B_n^\star > \frac{B_{n-1}}{2} > \frac{8^{n-1}}{2e\cdot 8!} = \frac{2^{n-5}}{2e\cdot 8!}\cdot 2^{2n+2} \geqslant \frac{2^{18}}{2e\cdot 8!}\cdot 2^{2n+2} > 2^{2n+2}. \qquad \dashv$$

Lemma 3.8 *In* $\mathcal{V}_{\mathrm{M}}$, $|\mathscr{P}(A)| < |\mathscr{B}(A)| < |\mathrm{Part}_{\mathrm{fin}}(A)| < |\mathrm{Part}(A)| < |\mathscr{P}(A^2)|$.

Proof. In $\mathcal{V}_{\mathrm{M}}$, since $\mathscr{P}(A^2)$ is Dedekind finite, and the injections constructed in the proofs of Facts 2.15 and 2.18 are clearly not surjective, it follows that $|\mathscr{B}_{\mathrm{fin}}(A)| < |\mathrm{Part}_{\mathrm{fin}}(A)| < |\mathrm{Part}(A)| < |\mathscr{P}(A^2)|$. By Lemma 3.4, it remains to show that $|\mathscr{P}(A)| < |\mathscr{B}(A)|$.

For a finite subset E of A, we can use $<_{\mathrm{M}}$ to define an ordering of the subsets of A supported by E and an ordering of the finitary partitions P of A with $\mathrm{ns}(P) \subseteq \mathscr{P}(E)$. Let $D = \{a_i \mid i < 46\}$ be a 46-element subset of A. In $\mathcal{V}_{\mathrm{M}}$, we define an injection f from $\mathscr{P}(A)$ into $\mathscr{B}(A)$ as follows.

Let $C \in \mathcal{V}_{\mathrm{M}}$ be a subset of A. By Lemma 3.5, C has a least support E. Suppose that C is the kth subset of A with E as its least support. Let $n = |D \triangle E|$, where $\triangle$ denotes the symmetric difference. Now, if $|E| \geqslant 23$, define $f(C)$ to be the kth finitary partition P of A with $\bigcup \mathrm{ns}(P) = E$; otherwise, define $f(C)$ to be the $(B_n^\star - k - 1)$th finitary partition P of A with $\bigcup \mathrm{ns}(P) = D \triangle E$. In the second case, $n \geqslant 23$, and thus $B_n^\star - k - 1 > 2^{2n+1}$ by Lemmas 3.6 and 3.7. Thus f is injective. Since D is a finite support of f, it follows that $f \in \mathcal{V}_{\mathrm{M}}$.

Finally, since $\mathscr{B}(A)$ is Dedekind finite and f is not surjective, it follows that $|\mathscr{P}(A)| < |\mathscr{B}(A)|$. $\dashv$

Now the next theorem immediately follows from Lemmas 3.4 and 3.8 and the Jech–Sochor theorem.

Theorem 3.9. *It is consistent with* ZF *that there exists an infinite set A for which* $\mathscr{B}(A) = \mathscr{B}_{\mathrm{fin}}(A)$ *and* $|\mathscr{P}(A)| < |\mathscr{B}(A)| < |\mathrm{Part}_{\mathrm{fin}}(A)| < |\mathrm{Part}(A)| < |\mathscr{P}(A^2)|$.

3.3. A model in which $\mathscr{B}(A)$ is incomparable with $\mathscr{P}(A)$ or $\mathrm{Part}_{\mathrm{fin}}(A)$. We use a variation of the basic Fraenkel model (cf. [4, pp. 195–196] or [6, Section 4.3]). Let A be an uncountable set of atoms, let $\mathcal{G}$ be the group of all permutations of A, and let $\mathcal{V}_{\mathrm{F}}$ be the permutation model determined by $\mathcal{G}$ and countable supports.

Lemma 3.10. *In* $\mathcal{V}_{\mathrm{F}}$, $|\mathscr{B}(A)| \nleqslant |\mathrm{Part}_{\mathrm{fin}}(A)|$, $|\mathscr{B}_{\mathrm{fin}}(A)| \nleqslant |\mathscr{P}(A)|$, *and* $|\mathscr{P}(A)| \nleqslant |\mathscr{B}(A)|$.

Proof. (1) $|\mathscr{B}(A)| \nleqslant |\mathrm{Part}_{\mathrm{fin}}(A)|$. Assume towards a contradiction that in $\mathcal{V}_{\mathrm{F}}$ there is an injection f from $\mathscr{B}(A)$ into $\mathrm{Part}_{\mathrm{fin}}(A)$. Let B be a countable support of f. Let $\{a_n \mid n \in \omega\} \subseteq A \setminus B$ with $a_i \neq a_j$ whenever $i \neq j$. Consider the finitary partition $P = \{\{a_{2n}, a_{2n+1}\} \mid n \in \omega\} \cup [A \setminus \{a_n \mid n \in \omega\}]^1 \in \mathcal{V}_{\mathrm{F}}$. Since $f(P)$ is a finite partition, there must be $i, j \in \omega$ with $i \neq j$ such that a_{2i} and a_{2j} are in the same block of $f(P)$. The transposition that swaps a_{2i} and a_{2i+1} fixes P, and thus also fixes $f(P)$, which implies that a_{2i+1} and a_{2j} are also in the same block of $f(P)$. Hence, the transposition that swaps a_{2i+1} and a_{2j} fixes $f(P)$, but it moves P, contradicting that f is injective.

(2) $|\mathscr{B}_{\mathrm{fin}}(A)| \nleqslant |\mathscr{P}(A)|$. Assume towards a contradiction that in $\mathcal{V}_{\mathrm{F}}$ there is an injection g from $\mathscr{B}_{\mathrm{fin}}(A)$ into $\mathscr{P}(A)$. Let C be a countable support of g. Let a_0, a_1, a_2, a_3 be four distinct elements of $A \setminus C$. Consider the finitary partition $P = \{\{a_0, a_1\}, \{a_2, a_3\}\} \cup [A \setminus \{a_0, a_1, a_2, a_3\}]^1 \in \mathcal{V}_{\mathrm{F}}$. Clearly, for any $i, j < 4$ with $i \neq j$, there is a $\pi \in \mathrm{fix}_{\mathcal{G}}(C)$ such that $\pi(P) = P$ and $\pi(a_i) = a_j$. Hence, $\{a_0, a_1, a_2, a_3\} \subseteq g(P)$ or $\{a_0, a_1, a_2, a_3\} \subseteq A \setminus g(P)$. Thus, the transposition that swaps a_0 and a_2 fixes $g(P)$, but it moves P, contradicting that g is injective.

(3) $|\mathscr{P}(A)| \nleqslant |\mathscr{B}(A)|$. Assume towards a contradiction that in $\mathcal{V}_{\mathrm{F}}$ there is an injection h from $\mathscr{P}(A)$ into $\mathscr{B}(A)$. Let D be a countable support of h. Let C be a denumerable subset of $A \setminus D$. Take $x \in C$ and $y \in A \setminus (C \cup D)$. We claim that $\{x\} \in h(C)$. Assume not; since $h(C)$ is finitary, we can find a $z \in C$ such that $z \notin [x]_{h(C)}$, and then the transposition that swaps x and z would fix C but move $h(C)$, which is a contradiction. Similarly, $\{y\} \in h(C)$. Hence, the transposition that swaps x and y fixes $h(C)$, but it moves C, contradicting that h is injective. ⊣

Now the next theorem immediately follows from Fact 2.19, Lemma 3.10, and the Jech–Sochor theorem.

THEOREM 3.11. *It is consistent with* ZF *that there exists an infinite set A such that*:

(i) $|\mathscr{B}(A)|$ *and* $|\mathrm{Part}_{\mathrm{fin}}(A)|$ *are incomparable*;
(ii) $|\mathscr{B}(A)|$ *and* $|\mathscr{P}(A)|$ *are incomparable*;
(iii) $|\mathscr{B}_{\mathrm{fin}}(A)|$ *and* $|\mathscr{P}(A)|$ *are incomparable.*

As easily seen, for infinite well-orderable A, $|A| = |\mathrm{fin}(A)| = |\mathscr{B}_{\mathrm{fin}}(A)|$ and $|\mathscr{P}(A)| = |\mathscr{B}(A)| = |\mathrm{Part}_{\mathrm{fin}}(A)| = |\mathrm{Part}(A)| = |\mathscr{P}(A^2)|$. Therefore, Theorems 3.3, 3.9, and 3.11 show that Figure 1 is optimal in the sense that the $\leqslant$, $<$, or $\neq$ relations between the cardinalities of these sets not indicted in the figure cannot be proved in ZF. However, we do not know whether $|\mathrm{Part}_{\mathrm{fin}}(A)| < |\mathscr{B}(A)|$ for some infinite set A is consistent with ZF.

§4. Theorems in ZF.

In this section, we prove in ZF some results concerning $\mathscr{B}(A)$, as well as the inequalities $|\mathscr{B}_{\mathrm{fin}}(A)| < |\mathrm{Part}_{\mathrm{fin}}(A)|$ and $|\mathscr{B}_{\mathrm{fin}}(A)| \neq |\mathscr{P}(A)|$ indicted in Figure 1.

Although Cantor's theorem may fail for $\mathscr{B}(A)$, it does hold under the existence of auxiliary functions.

DEFINITION 4.1. Let f be a function on A. An *auxiliary function* for f is a function g defined on $\mathrm{ran}(f)$ such that, for all $z \in \mathrm{ran}(f)$, $g(z)$ is a finitary partition of $f^{-1}[\{z\}]$ with at least one non-singleton block.

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$$\begin{tikzcd}
 & \mathscr{P}(A^2) & \\
 & \mathrm{Part}(A) \arrow[u] & \\
\mathscr{B}(A) \arrow[ur] & & \mathrm{Part}_{\mathrm{fin}}(A) \arrow[ul] \\
\mathscr{B}_{\mathrm{fin}}(A) \arrow[u] \arrow[urr, dashed] \arrow[rr, dashed, no head] & & \mathscr{P}(A) \arrow[u] \\
 & \mathrm{fin}(A) \arrow[ul] \arrow[uul, dashed] \arrow[ur, dashed] &
\end{tikzcd}$$

FIGURE 1. Relations between the cardinalities of some sets.

LEMMA 4.2. *From a function* $f : A \to \mathscr{B}(A)$ *and an auxiliary function for* f*, one can explicitly define a finitary partition of* A *not in* $\mathrm{ran}(f)$.

PROOF. Use Cantor's diagonal construction. Let f be a function from A to $\mathscr{B}(A)$ and let g be an auxiliary function for f. Let h be the function on $\mathrm{ran}(f)$ defined by

$$h(P) = \begin{cases} \{\{z\} \mid f(z) = P\}, & \text{if } x \sim_P y \text{ for some distinct } x, y \in f^{-1}[\{P\}], \\ g(P), & \text{otherwise.} \end{cases}$$

Then $\bigcup_{P \in \mathrm{ran}(f)} h(P)$ is a finitary partition of A not in $\mathrm{ran}(f)$. ⊣

4.1. A general result. We prove a general result which states that, if $\mathscr{B}(A)$ is Dedekind infinite, then there are no Dedekind finite-to-one functions from $\mathscr{B}(A)$ to $\mathrm{fin}(A)$.

LEMMA 4.3. *For any infinite ordinal* α*, one can explicitly define an injection* $f : \alpha \times \alpha \to \alpha$.

PROOF. See [18, 2.1]. ⊣

LEMMA 4.4. *From an infinite ordinal* α*, one can explicitly define an injection* $f : \mathrm{fin}(\alpha) \to \alpha$.

PROOF. See [4, Theorem 5.19]. ⊣

LEMMA 4.5. *From a finite-to-one function $f : \alpha \to A$, where α is an infinite ordinal, one can explicitly define an injection $g : \alpha \to A$.*

PROOF. See [12, Lemma 3.3]. ⊣

The main idea of the following proof is originally from [5, Theorem 3].

LEMMA 4.6. *From an injection $f : \alpha \to \operatorname{fin}(A)$, where α is an infinite ordinal, one can explicitly define an injection $h : \alpha \to \operatorname{fin}(A)$ such that the sets in $\operatorname{ran}(h)$ are pairwise disjoint and contain at least two elements.*

PROOF. Let f be an injection from α into $\operatorname{fin}(A)$, where α is an infinite ordinal. Let $\sim$ be the equivalence relation on A defined by

$$x \sim y \quad \text{if and only if} \quad \forall \beta < \alpha (x \in f(\beta) \leftrightarrow y \in f(\beta)).$$

Clearly, for every $x \in \bigcup \operatorname{ran}(f)$, the equivalence class $[x]_\sim$ is finite.

We want to show that there are α many such equivalence classes. In order to prove this, define a function Ψ on $\bigcup \operatorname{ran}(f)$ by

$$\Psi(x) = \left\{\gamma < \alpha \;\middle|\; x \in f(\gamma) \text{ and } \bigcap\{f(\beta) \mid \beta < \gamma \text{ and } x \in f(\beta)\} \nsubseteq f(\gamma)\right\}.$$

We claim that, for all $x, y \in \bigcup \operatorname{ran}(f)$,

$$x \sim y \quad \text{if and only if} \quad \Psi(x) = \Psi(y). \tag{1}$$

Clearly, if $x \sim y$ then $\Psi(x) = \Psi(y)$. For the other direction, assume towards a contradiction that $\Psi(x) = \Psi(y)$ but not $x \sim y$. Let $\delta < \alpha$ be the least ordinal such that $x \in f(\delta)$ is not equivalent to $y \in f(\delta)$. Without loss of generality, assume that $x \in f(\delta)$ but $y \notin f(\delta)$. Since $y \notin f(\delta)$, we have $\delta \notin \Psi(y) = \Psi(x)$, which implies that $\bigcap\{f(\beta) \mid \beta < \delta \text{ and } x \in f(\beta)\} \subseteq f(\delta)$. Since, for all $\beta < \delta$, $x \in f(\beta)$ if and only if $y \in f(\beta)$, it follows that $y \in \bigcap\{f(\beta) \mid \beta < \delta \text{ and } x \in f(\beta)\} \subseteq f(\delta)$, which is a contradiction.

We also claim that, for all $x \in \bigcup \operatorname{ran}(f)$,

$$\Psi(x) \in \operatorname{fin}(\alpha). \tag{2}$$

Let ξ be the least ordinal such that $x \in f(\xi)$. Then ξ is the first element of $\Psi(x)$. For all $\gamma, \delta \in \Psi(x)$ with $\gamma < \delta$, if $f(\xi) \cap f(\gamma) = f(\xi) \cap f(\delta)$, then $\bigcap\{f(\beta) \mid \beta < \delta \text{ and } x \in f(\beta)\} \subseteq f(\xi) \cap f(\gamma) \subseteq f(\delta)$, contradicting that $\delta \in \Psi(x)$. Hence, the function that maps each $\gamma \in \Psi(x)$ to $f(\xi) \cap f(\gamma)$ is an injection from $\Psi(x)$ into $\mathscr{P}(f(\xi))$. Since $\mathscr{P}(f(\xi))$ is finite, it follows that $\Psi(x)$ is finite.

Now, by Lemma 4.4, we can explicitly define an injection $p : \operatorname{fin}(\alpha) \to \alpha$. By (2), $\operatorname{ran}(\Psi) \subseteq \operatorname{fin}(\alpha)$. Let R be the well-ordering of $\operatorname{ran}(\Psi)$ induced by p; that is, $R = \{(a, b) \mid a, b \in \operatorname{ran}(\Psi) \text{ and } p(a) < p(b)\}$. Let θ be the order type of $\langle \operatorname{ran}(\Psi), R\rangle$, and let Θ be the unique isomorphism of $\langle \operatorname{ran}(\Psi), R\rangle$ onto $\langle \theta, \in\rangle$. It is easy to see that θ is an infinite ordinal.

Again, by Lemma 4.4, we can explicitly define an injection $q : \operatorname{fin}(\theta) \to \theta$. By (1), the function that maps each $\beta < \alpha$ to $\Psi[f(\beta)]$ is an injection from α into $\operatorname{fin}(\operatorname{ran}(\Psi))$. Let g be the function on α defined by

$$g(\beta) = \Theta^{-1}(q(\Theta[\Psi[f(\beta)]])).$$

g is visualized by the following diagram:

$$\begin{array}{ccccccccc} g: & \alpha & \to & \mathrm{fin}(\mathrm{ran}(\Psi)) & \to & \mathrm{fin}(\theta) & \to & \theta & \to & \mathrm{ran}(\Psi) \\ & \beta & \mapsto & \Psi[f(\beta)] & \mapsto & \Theta[\Psi[f(\beta)]] & \mapsto & q(\Theta[\Psi[f(\beta)]]) & \mapsto & g(\beta). \end{array}$$

Hence, g is an injection from α into $\mathrm{ran}(\Psi)$.

By Lemma 4.3, we can explicitly define an injection $s : \alpha \times \alpha \to \alpha$. Then the function h on α defined by

$$h(\beta) = \Psi^{-1}[\{g(s(\beta, 0)), g(s(\beta, 1))\}]$$

is the required function. ⊣

Corollary 4.7. *From an injection $f : \alpha \to \mathrm{fin}(A)$, where α is an infinite ordinal, one can explicitly define a surjection $g : A \twoheadrightarrow \alpha$ and an auxiliary function for g.*

Proof. Let f be an injection from α into $\mathrm{fin}(A)$, where α is an infinite ordinal. By Lemma 4.6, we can explicitly define an injection $h : \alpha \to \mathrm{fin}(A)$ such that the sets in $\mathrm{ran}(h)$ are pairwise disjoint and contain at least two elements. Now, the function g on A defined by

$$g(x) = \begin{cases} \text{the unique } \beta < \alpha \text{ for which } x \in h(\beta), & \text{if } x \in \bigcup \mathrm{ran}(h), \\ 0, & \text{otherwise}, \end{cases}$$

is a surjection from A onto α, and the function t on α defined by

$$t(\beta) = \begin{cases} \{h(0)\} \cup \{\{x\} \mid x \in A \setminus \bigcup \mathrm{ran}(h)\}, & \text{if } \beta = 0, \\ \{h(\beta)\}, & \text{otherwise}, \end{cases}$$

is an auxiliary function for g. ⊣

Now we are ready to prove our main theorem.

Theorem 4.8. *If $\mathscr{B}(A)$ is Dedekind infinite, then there are no Dedekind finite-to-one functions from $\mathscr{B}(A)$ to $\mathrm{fin}(A)$.*

Proof. Assume towards a contradiction that $\mathscr{B}(A)$ is Dedekind infinite and there is a Dedekind finite-to-one function $\Phi : \mathscr{B}(A) \to \mathrm{fin}(A)$. Let h be an injection from ω into $\mathscr{B}(A)$. In what follows, we get a contradiction by constructing by recursion an injection H from the proper class of ordinals into the set $\mathscr{B}(A)$.

For $n \in \omega$, take $H(n) = h(n)$. Now, we assume that α is an infinite ordinal and $H{\restriction}\alpha$ is an injection from α into $\mathscr{B}(A)$. Then $\Phi \circ (H{\restriction}\alpha)$ is a Dedekind finite-to-one function from α to $\mathrm{fin}(A)$. Since all Dedekind finite subsets of α are finite, $\Phi \circ (H{\restriction}\alpha)$ is finite-to-one. By Lemma 4.5, $\Phi \circ (H{\restriction}\alpha)$ explicitly provides an injection $f : \alpha \to \mathrm{fin}(A)$. Therefore, by Corollary 4.7, from f, we can explicitly define a surjection $g : A \twoheadrightarrow \alpha$ and an auxiliary function t for g. Then $(H{\restriction}\alpha) \circ g$ is a surjection from A onto $H[\alpha]$ and $t \circ (H{\restriction}\alpha)^{-1}$ is an auxiliary function for $(H{\restriction}\alpha) \circ g$. Hence, it follows from Lemma 4.2 that we can explicitly define an $H(\alpha) \in \mathscr{B}(A) \setminus H[\alpha]$ from $H{\restriction}\alpha$ (and Φ). ⊣

We draw some corollaries from the above theorem. The next two corollaries immediately follows from Theorem 4.8 and Facts 2.10, 2.11, and 2.17.

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Corollary 4.9. *If $\mathcal{B}(A)$ is Dedekind infinite, then there are no Dedekind finite-to-one functions from $\mathcal{B}(A)$ to* $\operatorname{seq}^{1\text{-}1}(A)$.

Corollary 4.10. *If $\mathcal{B}(A)$ is Dedekind infinite, then there are no Dedekind finite-to-one functions from $\mathcal{B}(A)$ to $\mathcal{B}_{\mathrm{fin}}(A)$, and thus $|\mathcal{B}_{\mathrm{fin}}(A)| < |\mathcal{B}(A)|$.*

Corollary 4.11. *If $\mathcal{B}(A) \neq \mathcal{B}_{\mathrm{fin}}(A)$, then $|\mathcal{B}_{\mathrm{fin}}(A)| < |\mathcal{B}(A)|$.*

Proof. Suppose $\mathcal{B}(A) \neq \mathcal{B}_{\mathrm{fin}}(A)$. Clearly, $|\mathcal{B}_{\mathrm{fin}}(A)| \leqslant |\mathcal{B}(A)|$. If $|\mathcal{B}(A)| = |\mathcal{B}_{\mathrm{fin}}(A)|$, since $\mathcal{B}_{\mathrm{fin}}(A) \subset \mathcal{B}(A)$, it follows that $\mathcal{B}(A)$ is Dedekind infinite, contradicting Corollary 4.10. Hence, $|\mathcal{B}_{\mathrm{fin}}(A)| < |\mathcal{B}(A)|$. ⊣

The following corollary is also proved in [10, Theorem 3.7].

Corollary 4.12. *For all infinite sets A, $|\mathrm{fin}(A)| < |\mathcal{B}(A)|$.*

Proof. By Fact 2.19, $|\mathrm{fin}(A)| \leqslant |\mathcal{B}_{\mathrm{fin}}(A)| \leqslant |\mathcal{B}(A)|$. If $|\mathcal{B}(A)| = |\mathrm{fin}(A)|$, since the injection constructed in the proof of Fact 2.19 is not surjective, it follows that $\mathcal{B}(A)$ is Dedekind infinite, contradicting Theorem 4.8. Thus, $|\mathrm{fin}(A)| < |\mathcal{B}(A)|$. ⊣

4.2. $|\mathcal{B}(A)| \neq |\operatorname{seq}(A)|$. We need the following result, which is a Kuratowski-like theorem for $\mathcal{B}(A)$.

Theorem 4.13. *For all sets A, the following are equivalent:*

(i) *$\mathcal{B}(A)$ is Dedekind infinite;*
(ii) *$\mathrm{ns}(P)$ is power Dedekind infinite for some $P \in \mathcal{B}(A)$;*
(iii) *$\mathcal{P}(\omega) \preccurlyeq \mathcal{B}(A)$.*

Proof. (i) ⇒ (ii). Suppose that $\mathcal{B}(A)$ is Dedekind infinite. Assume towards a contradiction that $\mathrm{ns}(P)$ is power Dedekind finite for all $P \in \mathcal{B}(A)$. Let $\langle P_k \mid k \in \omega\rangle$ be a denumerable family of finitary partitions of A. We define by recursion a sequence $\langle Q_n \mid n \in \omega\rangle$ of finitary partitions of A such that $\mathrm{ns}(Q_j) \neq \varnothing$ and $\bigcup \mathrm{ns}(Q_i) \cap \bigcup \mathrm{ns}(Q_j) = \varnothing$ whenever $i \neq j$ as follows.

Let $n \in \omega$ and assume that Q_i has already been defined for all $i < n$. Let $B = \bigcup_{i<n} \bigcup \mathrm{ns}(Q_i)$. By assumption, $\mathrm{ns}(Q_i)$ is power Dedekind finite, so is $\bigcup \mathrm{ns}(Q_i)$ by Fact 2.5. Hence, B is power Dedekind finite. Consider the following two cases:

Case 1. For a least $k \in \omega$, $x \sim_{P_k} y$ for some distinct $x, y \in A \setminus B$. Then we define

$$Q_n = \{[z]_{P_k} \setminus B \mid z \in A \setminus B\} \cup \{\{z\} \mid z \in B\}.$$

It is easy to see that Q_n is a finitary partition of A such that $\mathrm{ns}(Q_n) \neq \varnothing$ and $\bigcup \mathrm{ns}(Q_i) \cap \bigcup \mathrm{ns}(Q_n) = \varnothing$ for all $i < n$.

Case 2. Otherwise. Then, for all $k \in \omega$ and all $x \in \bigcup \mathrm{ns}(P_k) \setminus B$,

$$[x]_{P_k} \cap B \neq \varnothing \text{ and } [x]_{P_k} \setminus B = \{x\}. \tag{3}$$

Define by recursion a sequence $\langle k_m \mid m \in \omega\rangle$ of natural numbers as follows.

Let $m \in \omega$ and assume that k_j has been defined for $j < m$. By assumption, $\mathrm{ns}(P_{k_j})$ is power Dedekind finite, so is $\bigcup \mathrm{ns}(P_{k_j})$ by Fact 2.5. Therefore, $B \cup \bigcup_{j<m} \bigcup \mathrm{ns}(P_{k_j})$ is power Dedekind finite. Hence, there is a $k \in \omega$ such that $\mathrm{ns}(P_k) \nsubseteq \mathcal{P}(B \cup \bigcup_{j<m} \bigcup \mathrm{ns}(P_{k_j}))$. Define k_m to be the least such k. Then

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$$\bigcup \mathrm{ns}(P_{k_m}) \nsubseteq B \cup \bigcup_{j<m} \bigcup \mathrm{ns}(P_{k_j}). \tag{4}$$

Let f and g be the functions on ω defined by

$$f(m) = \bigcup \mathrm{ns}(P_{k_m}) \setminus \big(B \cup \bigcup_{j<m} \bigcup \mathrm{ns}(P_{k_j})\big),$$
$$g(m) = \{[x]_{P_{k_m}} \cap B \mid x \in f(m)\}.$$

By (3), for every $m \in \omega$, $g(m)$ is a finitary partition of a subset of B, and hence $\sim_{g(m)}$ is an element of $\mathscr{P}(B^2)$. Since $\mathscr{P}(B^2)$ is Dedekind finite by Fact 2.6, there are least $l_0, l_1 \in \omega$ with $l_0 < l_1$ such that $\sim_{g(l_0)} = \sim_{g(l_1)}$, and thus $g(l_0) = g(l_1)$. Let

$$D = \big\{\{x, y\} \bigm| x \in f(l_0), y \in f(l_1) \text{ and } [x]_{P_{k_{l_0}}} \cap B = [y]_{P_{k_{l_1}}} \cap B\big\}.$$

Since $l_0 < l_1$, $f(l_0) \cap f(l_1) = \varnothing$, and thus, by (3), the sets in D are pairwise disjoint. By (4), $f(m) \neq \varnothing$ for all $m \in \omega$, and since $g(l_0) = g(l_1)$, it follows that $D \neq \varnothing$. Now, we define

$$Q_n = D \cup \{\{z\} \mid z \in A \setminus \bigcup D\}.$$

Then Q_n is a finitary partition of A such that $\mathrm{ns}(Q_n) = D \neq \varnothing$; since $B \cap \bigcup D = \varnothing$, it follows that $\bigcup \mathrm{ns}(Q_i) \cap \bigcup \mathrm{ns}(Q_n) = \varnothing$ for all $i < n$.

Finally,

$$Q = \bigcup_{n\in\omega} \mathrm{ns}(Q_n) \cup \big\{\{z\} \mid z \in A \setminus \bigcup_{n\in\omega} \bigcup \mathrm{ns}(Q_n)\big\}$$

is a finitary partition of A such that $\mathrm{ns}(Q) = \bigcup_{n\in\omega} \mathrm{ns}(Q_n)$ is power Dedekind infinite, which is a contradiction.

(ii) $\Rightarrow$ (iii). Suppose that P is a finitary partition of A such that $\mathrm{ns}(P)$ is power Dedekind infinite. Let p be a surjection from $\mathrm{ns}(P)$ onto ω. Then the function h on $\mathscr{P}(\omega)$ defined by

$$h(u) = p^{-1}[u] \cup \{\{z\} \mid z \in A \setminus \bigcup p^{-1}[u]\}$$

is an injection from $\mathscr{P}(\omega)$ into $\mathscr{B}(A)$.

(iii) $\Rightarrow$ (i). Obviously. ⊣

Corollary 4.14. *If $\mathscr{B}(A)$ is Dedekind infinite, then there are no Dedekind finite-to-one functions from $\mathscr{B}(A)$ to* $\mathrm{seq}(A)$.

Proof. Assume towards a contradiction that $\mathscr{B}(A)$ is Dedekind infinite and there exists a Dedekind finite-to-one function from $\mathscr{B}(A)$ to $\mathrm{seq}(A)$. If A is Dedekind infinite, then $\mathrm{seq}(A) \approx \mathrm{seq}^{1\text{-}1}(A)$ by Fact 2.14, contradicting Corollary 4.9. Otherwise, by Fact 2.13, there is a Dedekind finite-to-one function from $\mathrm{seq}(A)$ to ω. By Theorem 4.13, $\mathscr{B}(\omega) \approx \mathscr{P}(\omega) \preccurlyeq \mathscr{B}(A)$, and therefore there is a Dedekind finite-to-one function from $\mathscr{B}(\omega)$ to ω, contradicting again Corollary 4.9. ⊣

Corollary 4.15. *For all non-empty sets A, $|\mathscr{B}(A)| \neq |\mathrm{seq}(A)|$.*

Proof. For all non-empty sets A, if $|\mathscr{B}(A)| = |\mathrm{seq}(A)|$, then it follows from Fact 2.12 that $\mathscr{B}(A)$ is Dedekind infinite, contradicting Corollary 4.14. ⊣

We do not know whether $|\mathscr{B}(A)| \neq |\mathrm{seq}^{1\text{-}1}(A)|$ for all infinite sets A is provable in ZF.

4.3. $|A^n| < |\mathscr{B}(A)|$**.** The main idea of the following proof is originally from [20, Lemma (i)] (cf. also [14]).

LEMMA 4.16. *For all* $n \in \omega$, *if* $|A| \geqslant 2n(2^{n+1}-1)$, *then* $|A^n| \leqslant |\mathscr{B}_{\mathrm{fin}}(A)|$.

PROOF. Let $n \in \omega$ and let A be a set with at least $2n(2^{n+1}-1)$ elements. Since $2^i = 1 + \sum_{k<i} 2^k$, we can choose $2n(2^{n+1}-1)$ distinct elements of A so that they are divided into $n+1$ sets H_i $(i \leqslant n)$ with

$$H_i = \{a_{i,j} \mid j < 2n\} \cup \{b_{i,x} \mid x \in H_k \text{ for some } k < i\}.$$

We construct an injection f from A^n into $\mathscr{B}_{\mathrm{fin}}(A)$ as follows. Without loss of generality, assume that $A \cap \omega = \varnothing$.

Let $s \in A^n$. Let i_s be the least $i \leqslant n$ for which $\mathrm{ran}(s) \cap H_i = \varnothing$. There is such an i because $|\mathrm{ran}(s)| \leqslant n$. Let t_s be the function on n defined by

$$t_s(j) = \begin{cases} s(j), & \text{if } s(j) \neq s(k) \text{ for all } k < j, \\ \max\{k < j \mid s(j) = s(k)\}, & \text{otherwise.} \end{cases}$$

Clearly, $t_s \in \mathrm{seq}^{1\text{-}1}(A \cup n)$. Let u_s be the function on n defined by

$$u_s(j) = \begin{cases} a_{i_s, n+t_s(j)}, & \text{if } t_s(j) \in n, \\ b_{i_s, t_s(j)}, & \text{if } t_s(j) \in H_k \text{ for some } k < i_s, \\ t_s(j), & \text{otherwise.} \end{cases}$$

Then it is easy to see that $u_s \in \mathrm{seq}^{1\text{-}1}(A)$. Now, we define

$$f(s) = \big\{\{a_{i_s,j}, u_s(j)\} \mid j < n\big\} \cup \big\{\{z\} \bigm| z \in A \setminus (\{a_{i_s,j} \mid j < n\} \cup \mathrm{ran}(u_s))\big\}.$$

Clearly, $f(s) \in \mathscr{B}_{\mathrm{fin}}(A)$. We prove that f is injective by showing that s is uniquely determined by $f(s)$ in the following way.

First, i_s is the least $i \leqslant n$ such that $H_i \cap \bigcup \mathrm{ns}(f(s)) \neq \varnothing$. Second, u_s is the function on n such that $\{a_{i_s,j}, u_s(j)\} \in f(s)$ for all $j < n$. Then, t_s is the function on n such that, for every $j < n$, either $t_s(j)$ is the unique element of n for which $u_s(j) = a_{i_s, n+t_s(j)}$, or $t_s(j)$ is the unique element of $\bigcup_{k<i_s} H_k$ for which $u_s(j) = b_{i_s, t_s(j)}$, or $t_s(j) = u_s(j) \notin H_{i_s}$. Finally, s is the function on n recursively determined by

$$s(j) = \begin{cases} t_s(j), & \text{if } t_s(j) \in A, \\ s(t_s(j)), & \text{otherwise.} \end{cases}$$

Hence, f is an injection from A^n into $\mathscr{B}_{\mathrm{fin}}(A)$. ⊣

COROLLARY 4.17. *For all* $n \in \omega$ *and all infinite sets* A, $|A^n| < |\mathscr{B}(A)|$.

PROOF. By Lemma 4.16, $|A^n| \leqslant |\mathscr{B}_{\mathrm{fin}}(A)| \leqslant |\mathscr{B}(A)|$. If $|\mathscr{B}(A)| = |A^n|$, since the injection constructed in the proof of Lemma 4.16 is not surjective, $\mathscr{B}(A)$ is Dedekind infinite, contradicting Corollary 4.14. Thus, $|A^n| < |\mathscr{B}(A)|$. ⊣

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LEMMA 4.18. *If A is Dedekind infinite, then* $|\mathrm{seq}(A)| \leqslant |\mathscr{B}_{\mathrm{fin}}(A)|$.

PROOF. Let h be an injection from ω into A. By the proof of Lemma 4.16, from $h{\restriction}(2n(2^{n+1}-1))$, we can explicitly define an injection

$$f_n : A^n \to \{P \in \mathscr{B}(A) \mid |\mathrm{ns}(P)| = n\}.$$

Then $\bigcup_{n\in\omega} f_n$ is an injection from $\mathrm{seq}(A)$ into $\mathscr{B}_{\mathrm{fin}}(A)$. ⊣

The following corollary immediately follows from Lemma 4.18 and Corollary 4.15.

COROLLARY 4.19. *For all Dedekind infinite sets A,* $|\mathrm{seq}(A)| < |\mathscr{B}(A)|$.

4.4. A Cantor-like theorem for $\mathscr{B}(A)$. Under the assumption that there is a finitary partition of A without singleton blocks, we show that Cantor's theorem holds for $\mathscr{B}(A)$. The key step of our proof is the following lemma.

LEMMA 4.20. *From a finitary partition P of A without singleton blocks and a surjection $f : A \twoheadrightarrow \alpha$, where α is an infinite ordinal, one can explicitly define a surjection $g : A \twoheadrightarrow \alpha$ and an auxiliary function for g.*

PROOF. Let P be a finitary partition of A without singleton blocks, and let f be a surjection from A onto α, where α is an infinite ordinal. Let

$$Q = \{f[E] \mid E \in P\}.$$

Clearly, $Q \subseteq \mathrm{fin}(\alpha)$. By Lemma 4.4, we can explicitly define an injection $p : \mathrm{fin}(\alpha) \to \alpha$. Let R be the well-ordering of Q induced by p; that is, $R = \{(a,b) \mid a,b \in Q \text{ and } p(a) < p(b)\}$. Since P is a partition of A, Q is a cover of α. Define a function h from α to Q by setting, for $\beta \in \alpha$,

$$h(\beta) = \text{the } R\text{-least } c \in Q \text{ such that } \beta \in c.$$

Since $\beta \in h(\beta)$ and $h(\beta)$ is finite for all $\beta \in \alpha$, h is a finite-to-one function from α to Q. Hence, by Lemma 4.5, h explicitly provides an injection from α into Q. Since $p{\restriction}Q$ is an injection from Q into α, it follows from Theorem 2.2 that we can explicitly define a bijection q between Q and α.

Now, the function g on A defined by

$$g(x) = q(f[[x]_P])$$

is a surjection from A onto α, and the function t on α defined by

$$t(\beta) = \{E \in P \mid q(f[E]) = \beta\}$$

is an auxiliary function for g. ⊣

THEOREM 4.21. *For all infinite sets A, if there is a finitary partition of A without singleton blocks, then there are no surjections from A onto $\mathscr{B}(A)$.*

PROOF. Let A be an infinite set and let P be a finitary partition of A without singleton blocks. Assume towards a contradiction that there is a surjection $\Phi : A \twoheadrightarrow \mathscr{B}(A)$. By Corollary 2.20, $\mathscr{B}(A)$ is power Dedekind infinite, so is A. Since $\bigcup \mathrm{ns}(P) = A$ is power Dedekind infinite, so is $\mathrm{ns}(P)$ by Fact 2.5. Hence, by Theorem 4.13, $\mathscr{B}(A)$ is Dedekind infinite. Let h be an injection from ω into $\mathscr{B}(A)$.

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In what follows, we get a contradiction by constructing by recursion an injection H from the proper class of ordinals into $\mathscr{B}(A)$.

For $n \in \omega$, take $H(n) = h(n)$. Now, we assume that α is an infinite ordinal and $H{\restriction}\alpha$ is an injection from α into $\mathscr{B}(A)$. Then $(H{\restriction}\alpha)^{-1} \circ \Phi$ is a surjection from a subset of A onto α and thus can be extended by zero to a surjection $f : A \twoheadrightarrow \alpha$. By Lemma 4.20, from P and f, we can explicitly define a surjection $g : A \twoheadrightarrow \alpha$ and an auxiliary function t for g. Then $(H{\restriction}\alpha) \circ g$ is a surjection from A onto $H[\alpha]$ and $t \circ (H{\restriction}\alpha)^{-1}$ is an auxiliary function for $(H{\restriction}\alpha) \circ g$. Hence, it follows from Lemma 4.2 that we can explicitly define an $H(\alpha) \in \mathscr{B}(A) \setminus H[\alpha]$ from $H{\restriction}\alpha$ (and P, Φ). ⊣

Under the same assumption, we also show that there are no finite-to-one functions from $\mathscr{B}(A)$ to A^n.

THEOREM 4.22. *For all infinite sets A, if there is a finitary partition of A without singleton blocks, then there are no finite-to-one functions from $\mathscr{B}(A)$ to A^n for every $n \in \omega$.*

PROOF. Let A be an infinite set and let P be a finitary partition of A without singleton blocks. Assume towards a contradiction that there is a finite-to-one function from $\mathscr{B}(A)$ to A^n for some $n \in \omega$. By Corollary 2.20, $\mathscr{B}(A)$ is power Dedekind infinite, so is A by Facts 2.5 and 2.6. Since $\bigcup \mathrm{ns}(P) = A$ is power Dedekind infinite, so is $\mathrm{ns}(P)$ by Fact 2.5. Hence, by Theorem 4.13, $\mathscr{B}(A)$ is Dedekind infinite, contradicting Corollary 4.14. ⊣

4.5. The inequalities $|\mathscr{B}_{\mathrm{fin}}(A)| < |\mathrm{Part}_{\mathrm{fin}}(A)|$ and $|\mathscr{B}_{\mathrm{fin}}(A)| \neq |\mathscr{P}(A)|$.

LEMMA 4.23. *If A is power Dedekind infinite, then there are no finite-to-one functions from $\mathscr{P}(A)$ to $\mathrm{fin}(A)$.*

PROOF. See [12, Corollary 3.7]. ⊣

The next corollary immediately follows from Lemma 4.23 and Facts 2.11 and 2.17.

COROLLARY 4.24. *If A is power Dedekind infinite, then there are no finite-to-one functions from $\mathscr{P}(A)$ to $\mathscr{B}_{\mathrm{fin}}(A)$.*

THEOREM 4.25. *For all infinite sets A, $|\mathscr{B}_{\mathrm{fin}}(A)| < |\mathrm{Part}_{\mathrm{fin}}(A)|$.*

PROOF. Let A be an infinite set. By Fact 2.18, $|\mathscr{B}_{\mathrm{fin}}(A)| \leqslant |\mathrm{Part}_{\mathrm{fin}}(A)|$. Assume $|\mathrm{Part}_{\mathrm{fin}}(A)| = |\mathscr{B}_{\mathrm{fin}}(A)|$. Since the injection constructed in the proof of Fact 2.18 is not surjective, $\mathrm{Part}_{\mathrm{fin}}(A)$ is Dedekind infinite, and thus A is power Dedekind infinite by Corollary 2.16. Now, by Fact 2.19, $|\mathscr{P}(A)| \leqslant |\mathrm{Part}_{\mathrm{fin}}(A)| = |\mathscr{B}_{\mathrm{fin}}(A)|$, contradicting Corollary 4.24. Hence, $|\mathscr{B}_{\mathrm{fin}}(A)| < |\mathrm{Part}_{\mathrm{fin}}(A)|$. ⊣

Finally, we prove $|\mathscr{B}_{\mathrm{fin}}(A)| \neq |\mathscr{P}(A)|$. For this, we need the following number-theoretic lemma.

LEMMA 4.26. *For each $n \in \omega$, let B_n be the nth Bell number; that is, $B_n = |\mathscr{B}(n)|$. Then B_m is not a power of 2 for all $m \geqslant 3$.*

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PROOF. Let $m \geqslant 3$. Then $B_m > 4$. It suffices to prove that B_m is not divisible by 8. By [7, Theorem 6.4], $B_{n+24} \equiv B_n (\text{mod } 8)$ for all $n \in \omega$. But B_n modulo 8 for n from 0 to 23 are

$$1, 1, 2, 5, 7, 4, 3, 5, 4, 3, 7, 2, 5, 5, 2, 1, 3, 4, 7, 1, 4, 7, 3, 2.$$

Hence, B_m is not divisible by 8. ⊣

LEMMA 4.27. *If* $\text{fin}(\mathscr{P}(A))$ *is Dedekind infinite, then* A *is power Dedekind infinite.*

PROOF. See [13, Theorem 3.2]. ⊣

THEOREM 4.28. *For all non-empty sets* A, $|\mathscr{P}(A)| \neq |\mathscr{B}_{\text{fin}}(A)|$.

PROOF. If A is a singleton, $|\mathscr{P}(A)| = 2 \neq 1 = |\mathscr{B}_{\text{fin}}(A)|$. Suppose $|A| \geqslant 2$, and fix two distinct elements a, b of A. Assume toward a contradiction that there is a bijection Φ between $\mathscr{P}(A)$ and $\mathscr{B}_{\text{fin}}(A)$. We define by recursion an injection f from ω into $\text{fin}(\mathscr{P}(A))$ as follows.

Take $f(0) = \{\{a\}, \{b\}\}$. Let $n \in \omega$, and assume that $f(0), \ldots, f(n)$ have been defined and are pairwise distinct elements of $\text{fin}(\mathscr{P}(A))$. Let $\sim$ be the equivalence relation on A defined by

$$x \sim y \quad \text{if and only if} \quad \forall C \in f(0) \cup \cdots \cup f(n)(x \in C \leftrightarrow y \in C).$$

Since $f(0), \ldots, f(n)$ are finite, the quotient set $A/{\sim}$ is a finite partition of A. Let $k = |A/{\sim}|$ and let $U = \{\bigcup W \mid W \subseteq A/{\sim}\}$. Then $|U| = 2^k$ and

$$f(0) \cup \cdots \cup f(n) \subseteq U. \tag{5}$$

Since $\{a\}, \{b\} \in A/{\sim}$, we have $k \geqslant 2$. Let $D = \bigcup\{\bigcup \text{ns}(P) \mid P \in \Phi[U]\}$. Since U is finite and $\Phi[U] \subseteq \mathscr{B}_{\text{fin}}(A)$, it follows that D is finite. Let $m = |D|$ and let $E = \{P \in \mathscr{B}_{\text{fin}}(A) \mid \bigcup \text{ns}(P) \subseteq D\}$. Then $|E| = B_m$ and $\Phi[U] \subseteq E$. Hence, $2^k = |U| = |\Phi[U]| \leqslant |E| = B_m$. Since $k \geqslant 2$, we have $m \geqslant 3$, which implies that $B_m \neq 2^k$ by Lemma 4.26, and hence $\Phi[U] \subset E$. Now, we define $f(n+1) = \Phi^{-1}[E \setminus \Phi[U]]$. Then $f(n+1)$ is a non-void finite subset of $\mathscr{P}(A)$. By (5), it follows that $f(n+1)$ is disjoint from each of $f(0), \ldots, f(n)$, and thus is distinct from each of them.

The existence of the above injection f shows that $\text{fin}(\mathscr{P}(A))$ is Dedekind infinite, which implies that, by Lemma 4.27, A is power Dedekind infinite, contradicting Corollary 4.24. ⊣

§5. Open questions.

We conclude the paper with the following five open questions.

QUESTION 5.1. *Are the following statements consistent with* ZF?

(1) *There exist an infinite set* A *and a finite-to-one function from* $\mathscr{B}(A)$ *to* A.
(2) *There exists an infinite set* A *for which* $|\mathscr{B}(A)| < |\mathcal{S}_3(A)|$, *where* $\mathcal{S}_3(A)$ *is the set of permutations of* A *with exactly three non-fixed points.*
(3) *There exists an infinite set* A *for which* $|\mathscr{B}(A)| = |\text{seq}^{1\text{-}1}(A)|$.
(4) *There exists an infinite set* A *for which* $|\text{Part}_{\text{fin}}(A)| < |\mathscr{B}(A)|$.
(5) *There exist an infinite set* A *and a surjection from* A^2 *onto* $\text{Part}(A)$.

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